

RESIDUE FIELDS FOR A CLASS OF RATIONAL \mathbf{E}_∞ -RINGS AND APPLICATIONS

AKHIL MATHEW

ABSTRACT. Let A be an \mathbf{E}_∞ -ring spectrum over the rational numbers. If A satisfies a noetherian condition on its homotopy groups $\pi_*(A)$, we construct a collection of \mathbf{E}_∞ - A -algebras that realize on homotopy the residue fields of $\pi_*(A)$. We prove an analog of the nilpotence theorem for these residue fields. As a result, we are able to give a complete algebraic description of the Galois theory of A and of the thick subcategories of perfect A -modules. We also obtain partial information on the Picard group of A .

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1. INTRODUCTION

1.1. Motivation. The goal of this paper is to describe certain invariants of structured ring spectra in characteristic zero. We start by first reviewing the motivation from stable homotopy theory.

The chromatic picture of stable homotopy theory identifies a class of “residue fields” which play an important role in global phenomena. Consider the following ring spectra:

- (1) $H\mathbb{Q}$: rational homology.
- (2) For each prime p , mod p homology $H\mathbb{F}_p$.
- (3) For each prime p and height n , the n th Morava K -theory $K(n)$.

These all define multiplicative homology theories on the category of spectra satisfying *perfect* Künneth isomorphisms: they behave like fields. Moreover, as a consequence of the deep nilpotence technology of [DHS88, HS98], they are powerful enough to describe much of the structure of the stable homotopy category. For example, one has the following result:

Theorem 1.1 (Hopkins-Smith [HS98]). *Let R be a ring spectrum and let $\alpha \in \pi_*(R)$. Then α is nilpotent if and only if the Hurewicz image of α in $\pi_*(F \otimes R)$ is nilpotent, as F ranges over all the ring spectra above.*

This fundamental result was used in [HS98] to classify the *thick subcategories* of the category of finite p -local spectra for a fixed prime p : all thick subcategories are defined by vanishing conditions for the various residue fields. One can attempt to ask such questions not only for spectra but for general symmetric monoidal, stable ∞ -categories, as Hovey, Palmieri, and Strickland have considered in [HPS97]; whenever one has an analog of Theorem 1.1, it is usually possible to prove results along these lines.

For instance, let A be an \mathbf{E}_∞ -ring. Then one can try to study such questions in the ∞ -category $\mathrm{Mod}(A)$ of A -modules. If $\pi_*(A)$ is concentrated in even degrees and is *regular noetherian*, then it is possible to construct residue fields, prove an analog of Theorem 1.1, and obtain a purely algebraic description of the thick subcategories of perfect A -modules. This has been observed independently by a number of authors (e.g., as a piece of forthcoming work of Antieau-Barthel-Gepner). For \mathbf{E}_∞ -rings (such as the \mathbf{E}_∞ -ring TMF of periodic topological modular forms) which are “built up” appropriately from such nice \mathbf{E}_∞ -rings, it is sometimes possible to construct residue fields as well. We used this to classify thick subcategories for perfect modules over \mathbf{E}_∞ -rings such as TMF in [Mat14b].

1.2. Statement of results. In this paper, we will study such questions over the *rational numbers*. Let A be a rational \mathbf{E}_∞ -ring such that the even homotopy groups $\pi_{\mathrm{even}}(A)$ form a noetherian ring and such that the odd homotopy groups $\pi_{\mathrm{odd}}(A)$ form a finitely generated $\pi_{\mathrm{even}}(A)$ -module. We will call such rational \mathbf{E}_∞ -rings *noetherian*.

For the statement of our first result, we work with \mathbf{E}_∞ -rings containing a unit in degree two. In this case, we will produce, for every prime ideal $\mathfrak{p} \subset \pi_0(A)$, a “residue field” of A , which will be an \mathbf{E}_∞ - A -algebra whose homotopy groups form a graded field.

Theorem 1.2. *Let A be a rational, noetherian \mathbf{E}_∞ -ring containing a unit in degree two. Given a prime ideal $\mathfrak{p} \subset \pi_0(A)$, there exists an \mathbf{E}_∞ - A -algebra $\kappa(\mathfrak{p})$ such that $\kappa(\mathfrak{p})$ is even periodic and the map $\pi_0(A) \rightarrow \pi_0(\kappa(\mathfrak{p}))$ induces the reduction $\pi_0 A \rightarrow \pi_0(A)_{\mathfrak{p}}/\mathfrak{p}\pi_0(A)_{\mathfrak{p}}$. $\kappa(\mathfrak{p})$ is unique up to homotopy as an object of the ∞ -category $\mathrm{CAlg}_{A/}$ of \mathbf{E}_∞ -rings under A .*

We will prove an analog of Theorem 1.1 in $\mathrm{Mod}(A)$ for these residue fields.

Theorem 1.3. *Suppose A is as above, and let B be an A -ring spectrum; that is, an algebra object in the homotopy category $\mathrm{Ho}(\mathrm{Mod}(A))$. Let $x \in \pi_*(B)$. Then x is nilpotent if and only if for every prime ideal $\mathfrak{p} \subset \pi_0(A)$, the image of x in $\pi_*(B \otimes_A \kappa(\mathfrak{p}))$ is nilpotent.*

The proof of Theorem 1.3 uses entirely different (and much simpler) techniques than Theorem 1.1. However, the conclusion is the same, and we thus find $\mathrm{Mod}(A)$ as an interesting new example of an “axiomatic stable homotopy theory” ([HPS97]) where many familiar techniques can be applied.

In particular, from Theorem 1.3, we will deduce a classification of thick subcategories of the ∞ -category $\mathrm{Mod}^\omega(A)$ of perfect A -modules, for A rational noetherian (not necessarily containing a unit in degree two). Let $\pi_{\mathrm{even}}(A) = \bigoplus_{i \in 2\mathbb{Z}} \pi_i(A)$; this is a graded ring, so $\mathrm{Spec} \pi_{\mathrm{even}}(A)$ inherits a \mathbb{G}_m -action.

Theorem 1.4. *Let A be a rational, noetherian \mathbf{E}_∞ -ring. The thick subcategories of $\mathrm{Mod}^\omega(A)$ are in natural correspondence with the subsets of the collection of homogeneous prime ideals of $\pi_{\mathrm{even}}(A)$ which are closed under specialization (equivalently, specialization-closed subsets of the stack $(\mathrm{Spec} \pi_{\mathrm{even}}(A))/\mathbb{G}_m$).*

We will then apply these ideas to the computation of Galois groups, which we introduced in [Mat14a] as an extension of Rognes’s work [Rog08]. The use of residue fields in Galois theory goes back to Baker-Richter’s work in [BR08], which studied the Galois groups of Morava E -theories at odd primes. We will show that the Galois theory of a noetherian rational \mathbf{E}_∞ -ring is “almost” entirely algebraic. (The “almost” comes from, e.g., the possibility of adjoining roots of periodicity generators in degrees $2n, n > 1$.) We prove:

Theorem 1.5. *If A is a noetherian rational \mathbf{E}_∞ -ring, then the Galois group of A is the étale fundamental group of the stack $(\mathrm{Spec} \pi_{\mathrm{even}}(A))/\mathbb{G}_m$.*

Finally, we will study the Picard groups of noetherian rational \mathbf{E}_∞ -rings. Here our results are much less conclusive, but we prove:

Theorem 1.6. *If A is a noetherian rational \mathbf{E}_∞ -ring, then the cokernel of the natural map $\mathrm{Pic}(\pi_*(A)) \rightarrow \mathrm{Pic}(A)$ (see Construction 7.4) is a torsion-free abelian group.*

Usually, results such as Theorem 1.4 and Theorem 1.5 are proved using strong homological assumptions on $\pi_*(A)$, e.g., that it is a regular ring. We will be able to get away with much weaker hypotheses on $\pi_*(A)$ (i.e., nothing close to regularity) because, over characteristic zero, \mathbf{E}_∞ -ring spectra are simpler. They have a more algebraic feel which gives one a wider range of techniques, and they have been studied in detail starting with Quillen's work on rational homotopy theory [Qui69]. In particular, there are two basic coincidences that will be used in this paper.

- (1) The free \mathbf{E}_∞ -ring on a generator in degree zero is equivalent to the suspension spectrum $\Sigma_+^\infty \mathbb{Z}_{\geq 0}$. In particular, as a result, it is possible to quotient an \mathbf{E}_∞ -ring by an element in degree zero to get a new \mathbf{E}_∞ -ring.
- (2) The free \mathbf{E}_∞ -ring on a generator in degree -1 is equivalent to cochains on S^1 . This has important descent-theoretic consequences and enables one to compare modules over this \mathbf{E}_∞ -ring with local systems on the circle S^1 .

Both these conditions are specific to the rational numbers. They fail away from characteristic zero, because of the existence of power operations [CLM76, BMMS86].

The above theorems rely crucially on the noetherianness hypotheses. We will discuss various counterexamples in §8. These counterexamples are related to classical purity questions for Picard groups and étale fundamental groups for local rings. As a consequence, we produce new examples of Galois extensions of ring spectra, which seems to be of interest in itself.

Organization. This paper is organized as follows. In §2, we analyze the operation of attaching a 1-cell in a rational \mathbf{E}_∞ -ring in detail. In §3, we do the same for the operation of attaching a 0-cell, via a comparison between modules over the cochain \mathbf{E}_∞ -ring $C^*(S^1; \mathbb{Q})$ and local systems on S^1 . The main technical results (existence of the residue fields and the nilpotence theorem) are proved in §4. §5, §6, and §7 contain the applications to thick subcategories, Galois groups, and Picard groups, respectively. Finally, §8 discusses various non-noetherian counterexamples.

Notation. In this paper, we will adopt the following notational conventions. We will frequently identify abelian groups with their Eilenberg-MacLane spectra without additional notation. The letters R, S, T, \dots will refer to ordinary (discrete) rings. The subscript $*$ will refer to a grading. We will let \mathbf{CAlg} denote the ∞ -category of \mathbf{E}_∞ -rings. The letters A, B, C will refer to \mathbf{E}_∞ -rings. Given an \mathbf{E}_∞ -ring A , we let $\mathbf{Mod}(A)$ denote the ∞ -category of A -modules and $\mathbf{Mod}^\omega(A) \subset \mathbf{Mod}(A)$ the full subcategory spanned by the perfect A -modules. If X is a space and A an \mathbf{E}_∞ -ring, we let $C^*(X; A)$ denote the cochain \mathbf{E}_∞ -algebra on X with values in A , often also denoted A^X or $\mathbf{Fun}(X_+, A)$. Finally, if $A \in \mathbf{CAlg}$ is a *rational* \mathbf{E}_∞ -ring, we will write $A[t_2^{\pm 1}]$ for the free \mathbf{E}_∞ - A -algebra on an invertible degree two generator.

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2. DEGREE ZERO ELEMENTS OF RATIONAL \mathbf{E}_∞ -RINGS

In this section, we describe the first set of the basic characteristic zero techniques needed for this paper. In particular, we discuss the “coincidence” of \mathbf{E}_∞ -rings that $\Sigma_+^\infty \mathbb{Z}_{\geq 0}$ is free on a degree zero class and thus analyze the operation of attaching cells in degree one. The main result of the section (Theorem 2.22) controls the behavior on homotopy rings of attaching cells in degree one. We also prove a version (Proposition 2.15) of the classical result that a complete local ring with residue field of characteristic zero contains a copy of its residue field.

Some of the more refined results require the noetherianness hypothesis that will be crucial for most of the main results of this paper.

Definition 2.1. We say that a rational \mathbf{E}_∞ -ring A is *noetherian* if:

- (1) The commutative ring $\pi_{\text{even}}(A)$ is noetherian.
- (2) The $\pi_{\text{even}}(A)$ -module $\pi_{\text{odd}}(A)$ is finitely generated.

The “noetherian” hypothesis ensures that certain categorical constructions one may perform on A affect the homotopy groups of A in a reasonable manner and, as such, will be indispensable to this paper.

Warning 2.2. The noetherian condition is not a purely categorical one. For instance, a finitely presented \mathbf{E}_∞ -algebra over a noetherian \mathbf{E}_∞ -ring (even \mathbb{Q}) need not be noetherian, i.e., the analog of Hilbert’s basis theorem fails. See Section 8.2 for an example.

2.1. Cofibers of degree zero elements. Let A be an \mathbf{E}_∞ -ring and let $x \in \pi_k(A)$, defining a map of A -modules $\Sigma^k A \xrightarrow{x} A$.

Definition 2.3. We will write A/x for the cofiber of this map $x: \Sigma^k A \rightarrow A$.

One wants to think of A/x as a homotopy-theoretic “quotient” of A by the “ideal” generated by x and, as in algebra, turn this into an \mathbf{E}_∞ -ring under A . There is, in general, no reason to expect this to be possible (or canonical in any way).

Example 2.4. The sphere S^0 is the most basic example of an \mathbf{E}_∞ -ring, but it is folklore that the Moore spectrum $S^0/2$ cannot even be a ring spectrum up to homotopy.

The obstructions to multiplicative structures have been discussed, for example, in [Oka79, Str99]. Some further obstructions to *structured* multiplications, via the theory of power operations, are discussed in [MNN14].

Suppose first that $k = 0$. To understand this failure, recall how the quotient is constructed in classical commutative algebra. Let R be a (classical) commutative ring, and fix $x \in R$. The (classical) quotient $R/(x)$ is the pushout of the diagram of commutative rings

$$\begin{array}{ccc} \mathbb{Z}[t] & \xrightarrow{t \mapsto 0} & \mathbb{Z} \\ \downarrow x \mapsto t & & \downarrow \\ R & \longrightarrow & R/(x) \end{array}.$$

Here $\mathbb{Z}[t]$ is the free commutative ring on a generator t , and forming the pushout $R/(x)$ as above amounts to setting $x = 0$.

In homotopy theory, one can make a similar construction, which has been discussed in [Szy13].

Definition 2.5. There is a free \mathbf{E}_∞ -ring on a single generator, denoted $S^0\{t\}$, whose underlying spectrum is given by

$$S^0\{t\} \simeq \bigoplus_{n \geq 0} \Sigma_+^\infty B\Sigma_n,$$

as $\{\Sigma_n\}_{n \geq 0}$ ranges over the symmetric groups. Given an \mathbf{E}_∞ -ring A and an element $x \in \pi_0 A$, we obtain a map of \mathbf{E}_∞ -rings $S^0\{t\} \rightarrow A$ sending $t \mapsto x$, by the universal property: the space of maps of \mathbf{E}_∞ -rings $S^0\{t\} \rightarrow A$ is precisely $\Omega^\infty A$. In particular, we can form a pushout square in \mathbf{CAlg} ,

$$\begin{array}{ccc} S^0\{t\} & \xrightarrow{t \mapsto 0} & S^0 \\ \downarrow t \mapsto x & & \downarrow \\ A & \longrightarrow & A' \end{array}$$

where A' is called the *free A -algebra with $x = 0$* . Following Szymik [Szy13], we will write $A' = A//x$.

Given an \mathbf{E}_∞ - A -algebra A'' , the space $\mathrm{Hom}_{A//x}(A', A'')$ is the space of nullhomotopies of x in A'' (which is empty unless x maps to zero in $\pi_0 A''$, and in this case is $\Omega^{\infty+1} A''$).

For future reference, it will be convenient to have the following more general definition.

Definition 2.6. For X a spectrum, we write $\mathrm{Sym}^*(X)$ for the free \mathbf{E}_∞ -ring on X , so that $\mathrm{Sym}^*(X) \simeq \bigoplus_{n \geq 0} (X^{\otimes n})_{h\Sigma_n}$. If A is an \mathbf{E}_∞ -ring and $x \in \pi_k A$ is an element, we denote by

$A//x$ the pushout

$$\begin{array}{ccc} \mathrm{Sym}^*(S^k) & \xrightarrow{x} & A \\ \downarrow 0 & & \downarrow \\ S^0 & \longrightarrow & A//x \end{array},$$

where the map $\mathrm{Sym}^*(S^k) \rightarrow A$ is determined by the map $x: S^k \rightarrow A$, and where $\mathrm{Sym}^*(S^k) \rightarrow S^0$ is determined by $0: S^k \rightarrow S^0$. We will call $A//x$ the *free \mathbf{E}_∞ - A -algebra with $x = 0$* . Given a sequence of elements $x_1, \dots, x_n \in \pi_*(A)$, we will write $A//x_1, \dots, x_n$ for the iterated quotient $(\dots (A//x_1)//x_2) \dots //x_{n-1}) \dots //x_n$.

We return to the case $k = 0$. In general, if $x \in \pi_0 A$ is fixed, then Definition 2.5 gives

$$A//x \simeq A \otimes_{S^0\{t\}} S^0,$$

since pushouts of \mathbf{E}_∞ -rings are relative tensor products. This is usually very different, as an A -module, from A/x . For example, the free \mathbf{E}_∞ -ring with $p^n = 0$ is not S^0/p^n . From the “chromatic” point of view, it is actually invisible: its E_r -localization vanishes for each r , by the main result of [MNN14].

Remark 2.7. In fact, the $S^0\{t\}$ -module S^0 is quite complicated, and is not, for example, perfect. For instance, if we worked over \mathbb{F}_2 rather than S^0 , then $\mathbb{F}_2\{t\}$ has homotopy groups given by a polynomial ring on the tautological class t and certain admissible monomials in the Dyer-Lashof algebra applied to t ([CLM76]), so that \mathbb{F}_2 is an infinite quotient of $\mathbb{F}_2\{t\}$ by a regular sequence of polynomial generators.

However, there is another \mathbf{E}_∞ -ring which is better behaved in this regard, and which *will* enable us to place \mathbf{E}_∞ -structures on quotients in certain cases. Recall that the \mathbf{E}_∞ -ring $S^0\{t\}$ is obtained from the free \mathbf{E}_∞ -space on a single generator by applying Σ_+^∞ . This is the free symmetric monoidal category on one object: the groupoid of finite sets and isomorphisms between them, or topologically $\bigsqcup_{n \geq 0} B\Sigma_n$.

Definition 2.8. We can apply Σ_+^∞ instead to the symmetric monoidal groupoid $\mathbb{Z}_{\geq 0}$, which has objects given by the natural numbers (under addition) and no nontrivial isomorphisms. The resulting \mathbf{E}_∞ -ring $\Sigma_+^\infty \mathbb{Z}_{\geq 0}$, the “monoid algebra” of the natural numbers (as studied in [ABG⁺14]), will be written $S^0[t]$ since its homotopy groups actually are given by $(\pi_* S^0)[t]$. More generally, we will write $A[t]$ for $A \otimes \Sigma_+^\infty \mathbb{Z}_{\geq 0}$, if A is any \mathbf{E}_∞ -ring.

Construction 2.9. Now let A be an \mathbf{E}_∞ -ring, and let $S^0\{t\} \rightarrow A$ be a map classifying an element $x \in \pi_0(A)$. Suppose that we have a factorization in the ∞ -category \mathbf{CAlg}

$$\begin{array}{ccc} S^0\{t\} & \xrightarrow{x} & A \\ \downarrow & \nearrow & \\ S^0[t] & & \end{array}$$

In this case, we can form the relative tensor product $A \otimes_{S^0[t]} S^0$ (using the map $S^0[t] \rightarrow S^0$ which sends $t \mapsto 0$), as an \mathbf{E}_∞ -ring. The cofiber sequence of $S^0[t]$ -modules

$$S^0[t] \xrightarrow{t} S^0[t] \rightarrow S^0,$$

shows that this relative tensor product, as an A -module spectrum, is actually A/x .

In other words, by the universal property of the monoid algebra, if there exists a factorization in the diagram of \mathbf{E}_∞ -spaces

$$\begin{array}{ccc} \bigsqcup_{n \geq 0} B\Sigma_n & \xrightarrow{x} & \Omega^\infty A \\ \downarrow & \nearrow & \\ \mathbb{Z}_{\geq 0} & & \end{array}$$

where $\Omega^\infty A$ is given the *multiplicative* \mathbf{E}_∞ -structure, then we can place a *natural* \mathbf{E}_∞ -structure on A/x .

Remark 2.10. Let X be an \mathbf{E}_∞ -space and let $x \in \pi_0 X$, classified by a map of \mathbf{E}_∞ -spaces $\bigsqcup_{n \geq 0} B\Sigma_n \rightarrow X$. If this map admits a factorization over $\mathbb{Z}_{\geq 0}$, then x has been called by Lurie a “strictly commutative” element of X . Construction 2.9 shows that, while arbitrary cofibers A/x need not admit \mathbf{E}_∞ -structures, one can find an \mathbf{E}_∞ -structure if x is strictly commutative.

Unfortunately, in general, describing maps out of $S^0[t]$ is difficult, since $\mathbb{Z}_{\geq 0}$ does not admit a simple presentation as an \mathbf{E}_∞ -space. In *characteristic zero*, i.e., over \mathbb{Q} , the natural map

$$\mathbb{Q}\{t\} \rightarrow \mathbb{Q}[t],$$

becomes an equivalence of \mathbf{E}_∞ -rings, because the maps $(B\Sigma_n)_+ \rightarrow S^0$ are rational equivalences. In particular, given any rational \mathbf{E}_∞ -ring A , and an element $x \in \pi_0(A)$, we can obtain a map

$$\mathbb{Q}[t] \rightarrow A, \quad t \mapsto x,$$

and we can form the relative tensor product $A/x \simeq A \otimes_{\mathbb{Q}[t]} \mathbb{Q}$ as an \mathbf{E}_∞ -ring. This process can equivalently be described as attaching a 1-cell to kill the element $x \in \pi_0 A$, i.e., as forming $A//x$. We may summarize these observations in the following proposition.

Proposition 2.11. *Let A be a rational \mathbf{E}_∞ -ring and let $x \in \pi_0(A)$. Then $A//x \in \mathrm{CAlg}_{A/}$ has as underlying A -module A/x . In particular, we may make A/x into an \mathbf{E}_∞ - A -algebra.*

Remark 2.12. Forthcoming work of Hopkins-Lurie has shown that the \mathbf{E}_∞ -ring spectra $S^0[t]$ are actually tractable if one works under Morava E -theory E_n , and $K(n)$ -locally. For example, if one works $K(1)$ -locally and under p -adic K -theory \widehat{KU}_p , then $L_{K(1)}(\widehat{KU}_p[t])$ has a decomposition as a two-cell complex, obtained by starting with the free \mathbf{E}_∞ -ring on one generator t and then killing $\theta(t)$, where θ is the basic power operation for $K(1)$ -local \mathbf{E}_∞ -rings. This can be deduced from the analysis of $K(1)$ -local power operations in [Hop]. In particular, in their forthcoming work, they have been able to describe the space of maps $S^0[t^{\pm 1}] \rightarrow E_n$.

It is similarly possible to quotient by *even degree* elements of a rational \mathbf{E}_∞ -ring, as we show below. In fact, we did not strictly need the discussion of “strict commutativity” for the present paper, but included it for its intrinsic interest (as it becomes more relevant away from characteristic zero).

Proposition 2.13. *Let A be a rational \mathbf{E}_∞ -ring and let $x \in \pi_n(A)$. Suppose n is an even integer. Then $A//x \in \mathrm{CAlg}_{A/}$ has underlying A -module A/x .*

Proof. Recall that $\pi_* \mathrm{Sym}^*(\mathbb{Q}[n])$ is a polynomial ring on a class in degree n . Thus, the result follows from $A//x \simeq A \otimes_{\mathrm{Sym}^*(\mathbb{Q}[n])} \mathbb{Q}$ and the cofiber sequence $\Sigma^n \mathrm{Sym}^* \mathbb{Q}[n] \rightarrow \mathrm{Sym}^* \mathbb{Q}[n] \rightarrow \mathbb{Q}$ of $\mathrm{Sym}^* \mathbb{Q}[n]$ -modules. \square

2.2. The Cohen structure theorem. Let (R, \mathfrak{m}) be a complete local noetherian ring with residue field k of characteristic zero. In this case, a basic piece of the Cohen structure theorem (see for instance [Eis95, Ch. 8]) implies that R contains a copy of its residue field:

Theorem 2.14 (Cohen). *Hypotheses as above, the projection $R \rightarrow R/\mathfrak{m} \rightarrow k$ admits a section.*

We refer to [Mat80, Theorem 60, §28.J] for a proof of a more general result than Theorem 2.14. It is closely related to the fact that, in characteristic zero, all field extensions can be obtained as an inductive limit of smooth morphisms; this argument implies an analogous result in the world of \mathbf{E}_∞ -ring spectra, and it is the purpose of this subsection to describe that. In particular, we prove:

Proposition 2.15. *Let A be a noetherian, rational \mathbf{E}_∞ -ring such that $\pi_0 A$ is a complete local ring with residue field k . Then there exists a morphism of \mathbf{E}_∞ -ring spectra $k \rightarrow A$ such that on π_0 , the composite map $k \rightarrow \pi_0 A \rightarrow k$ is the identity.*

Proof. By Theorem 2.14, there is a section $\phi: k \rightarrow \pi_0 A$ of the reduction map. We want to realize this topologically. To start with, there is a (unique) map $\mathbb{Q} \rightarrow A$ as A is rational. Let $\{t_\alpha\}_{\alpha \in \Gamma}$ be a transcendence basis of k/\mathbb{Q} , so that we have field extensions

$$\mathbb{Q} \subset \mathbb{Q}(\{t_\alpha\}) \subset k,$$

where the first extension is purely transcendental and the second extension is algebraic. For each $\alpha \in \Gamma$, choose $u_\alpha \in \pi_0 A$ be defined by $u_\alpha = \phi(t_\alpha)$, so that u_α projects to t_α in the residue field. We obtain a map of \mathbf{E}_∞ -ring spectra

$$\bigotimes_{\Gamma} \mathbb{Q}[t_\alpha] \rightarrow A, \quad t_\alpha \mapsto u_\alpha,$$

where the left-hand-side is a free \mathbf{E}_∞ -ring on $|\Gamma|$ variables (i.e., a discrete polynomial ring on the $\{t_\alpha\}$). It necessarily factors over the localization $\mathbb{Q}(\{t_\alpha\})$, so we obtain a map $\mathbb{Q}(\{t_\alpha\}) \rightarrow A$. This realizes on π_0 the restriction $\phi|_{\mathbb{Q}(\{t_\alpha\})}$.

Finally, we want to find an extension over k such that the diagram

$$\begin{array}{ccc} \mathbb{Q}(\{t_\alpha\}) & \xrightarrow{\quad} & k \\ \downarrow & \swarrow & \\ A & & \end{array}$$

such that the composite $k \rightarrow \pi_0 A \rightarrow k$ is the identity. Since k is a colimit of finite *étale* $\mathbb{Q}(\{t_\alpha\})$ -algebras (i.e., finite separable extensions), it is equivalent to doing this at the level of π_0 ([Lur14, §7.5]), and the map $\phi: k \rightarrow \pi_0(A)$ enables us to do that. \square

Remark 2.16. The argument shows that the set of *homotopy classes* of maps of \mathbf{E}_∞ -rings $k \rightarrow A$ is in bijection with the set of ring-homomorphisms $k \rightarrow \pi_0(A)$.

2.3. Properties of the quotient \mathbf{E}_∞ -ring. In order to make the ideas sketched above work, we will need a few more preliminaries on Construction 2.5 and its behavior on homotopy. If A is a rational \mathbf{E}_∞ -ring and $x \in \pi_0 A$, then the homotopy groups of $A//x \simeq A/x$ are determined additively by the short exact sequence

$$(1) \quad 0 \rightarrow \pi_j(A)/x\pi_j(A) \rightarrow \pi_j(A/x) \rightarrow (\ker x)|_{\pi_{j-1}(A)} \rightarrow 0,$$

but this fails to determine the precise multiplicative structure. In this section, we will show (Theorem 2.22) that the multiplicative structure is not so different from that of the subring $\pi_*(A)/x\pi_*(A)$ under noetherian hypotheses. We will not need the full strength of these results in the sequel.

We begin by reviewing the theory of finite universal homeomorphisms. Recall that a map of rings $R \rightarrow R'$ is called a *universal homeomorphism* if, for every R -algebra R'' , the map $R'' \rightarrow R'' \otimes_R R'$ induces a homeomorphism upon applying Spec .

Proposition 2.17. *A morphism $R \rightarrow R'$, such that R' is a finitely generated R -module, is a universal homeomorphism if and only if, for every morphism $R \rightarrow k$ where k is a field, the base-change $R' \otimes_R k$ is a local artinian k -algebra (in particular, nonzero).*

Proof. Suppose that for every map from R to a field k , the base-change $R' \otimes_R k$ is a local artinian k -algebra. It follows that the residue field of $R' \otimes_R k$ is necessarily a purely inseparable extension of k , since otherwise we can replace k by \bar{k} and $R' \otimes_R \bar{k}$ would have nontrivial idempotents. It then follows that the map $\text{Spec} R' \rightarrow \text{Spec} R$ is a radicial morphism [Sta13, Tag 0480] and thus in particular universally injective. Given any R -algebra R'' , the map $R'' \rightarrow R' \otimes_R R''$ is finite, so induces a closed map on Spec which is also injective since $R \rightarrow R'$ is radicial. The map is surjective since all the fibers are nonempty by assumption, and thus a homeomorphism.

Conversely, suppose $R \rightarrow R'$ is a finite universal homeomorphism. Then all the base-changes $k \rightarrow R' \otimes_R k$, for an R -field k , are universal homeomorphisms themselves. In particular, $\text{Spec} R' \otimes_R k$ is connected. But $R' \otimes_R k$ is a finite-dimensional k -algebra, so if its spectrum is connected, then $R' \otimes_R k$ must be local artinian. \square

Corollary 2.18. *A finite map $R \rightarrow R'$ of \mathbb{Q} -algebras is a universal homeomorphism if and only if for every residue field $R \rightarrow k$, the tensor product $R' \otimes_R k$ is local with residue field k .*

Proof. This follows from the fact that all finite extensions in characteristic zero are separable, so that if A is a finite-dimensional local k -algebra with residue field strictly containing k , then $A \otimes_k \bar{k}$ necessarily has nontrivial idempotents. \square

We will now begin working towards the proof of Theorem 2.22. We will first need a preliminary lemma on idempotents in these quotients.

Definition 2.19. If A is an \mathbf{E}_∞ -ring, we let $\text{Idem}(A)$ denote the set of idempotents in $\pi_0 A$. The construction $A \mapsto \text{Idem}(A)$ sends homotopy limits of \mathbf{E}_∞ -rings to inverse limits of sets, since the set $\text{Idem}(A)$ is homotopy equivalent to the space of maps of \mathbf{E}_∞ -rings $S^0 \times S^0 \rightarrow A$ in view of the theory of étale algebras [Lur14, §7.5]. If R is a discrete ring, we will also write $\text{Idem}(R)$ for the set of idempotents in R .

Lemma 2.20. *Let A be a rational noetherian \mathbf{E}_∞ -ring with $\pi_0(A)$ local and let $x_1, \dots, x_r \in \pi_0(A)$. Then the map $\pi_0(A)/(x_1, \dots, x_r) \rightarrow \pi_0(A// (x_1, \dots, x_r))$ of discrete rings induces an isomorphism on Idem .*

Proof. In fact, we have a map of \mathbf{E}_∞ -rings $A \rightarrow A// (x_1, \dots, x_r)$, and if we form the cobar construction on this map, we obtain a coaugmented cosimplicial object

$$A// (x_1, \dots, x_r) \rightrightarrows A// (x_1, \dots, x_r) \otimes_A A// (x_1, \dots, x_r) \rightrightarrows \dots,$$

whose homotopy limit is the (x_1, \dots, x_r) -adic completion of A . We refer to [Lur11b, §4] for preliminaries on completions of ring spectra. In particular, the idempotents in the totalization are the same as the idempotents in the (x_1, \dots, x_r) -adic completion of $\pi_0 A$, or equivalently, by the lifting idempotents theorem [Eis95, Cor. 7.5], in $\pi_0(A)/(x_1, \dots, x_r)$.

Since the operation of taking idempotents commutes with homotopy limits, we conclude that the set of idempotents in $\pi_0(A)/(x_1, \dots, x_r)$ is the reflexive equalizer

$$\text{Idem}(A// (x_1, \dots, x_r)) \rightrightarrows \text{Idem}(A// (x_1, \dots, x_r) \otimes_A A// (x_1, \dots, x_r)).$$

However, we claim that the two maps in the reflexive equalizers are isomorphisms (and thus equal). In fact, $A// (x_1, \dots, x_r) \otimes_A A// (x_1, \dots, x_r)$ is obtained by attaching 1-cells to kill the classes x_1, \dots, x_r in $A// (x_1, \dots, x_r)$ which are *already zero*; in particular, as an \mathbf{E}_∞ -ring, we have

$$A// (x_1, \dots, x_r) \otimes_A A// (x_1, \dots, x_r) \simeq A// (x_1, \dots, x_r) \otimes \text{Sym}^*[y_1, \dots, y_r], \quad |y_i| = 1,$$

which has the same idempotents as $A// (x_1, \dots, x_r)$. \square

Proposition 2.21. *Let A be a rational noetherian \mathbf{E}_∞ -ring and let $x \in \pi_0(A)$. The map $\pi_0(A)/(x) \rightarrow \pi_0(A// x)$ is a finite universal homeomorphism.*

Proof. We already know that $\pi_0(A// x)$ is a finitely generated $\pi_0(A)/(x)$ -module, by the short exact sequence (1). We will check that the map is a finite universal homeomorphism fiberwise at each prime.

Fix a prime ideal \mathfrak{p} of $\pi_0(A)/(x)$. Let $x_1, \dots, x_n \in \pi_0(A)$ project to generators of \mathfrak{p} . Localizing A at \mathfrak{p} , we may assume that $\pi_0(A)$ is local with maximal ideal \mathfrak{p} . By completing A , we may assume that A admits the structure of an \mathbf{E}_∞ - k -algebra for k the residue field of $\pi_0(A)$, in view of Proposition 2.15.

We need to show that the map of commutative rings

$$(2) \quad \pi_0(A)/(x, x_1, \dots, x_n) \rightarrow \pi_0(A// x)/(x_1, \dots, x_n)$$

is a finite universal homeomorphism. The left-hand-side of (2) is the residue field k of $\pi_0(A)$ at the maximal ideal, and the right-hand-side is a finite module over the left-hand-side and is in particular a product of local artinian k -algebras. By replacing A with $A \otimes_k k'$ for a finite extension k'/k , we may assume that each of the residue fields of the left-hand-side of (2) is k itself. Thus, it suffices to show $\pi_0(A// x)/(x_1, \dots, x_n)$ has no nontrivial idempotents in this case. As a result, our claim follows from Lemma 2.20, which implies that the connected components of $\text{Spec} \pi_0(A// x)/(x_1, \dots, x_n)$ are in bijection with those of $\text{Spec} \pi_0(A// (x, x_1, \dots, x_n))$, and in turn with those of $\text{Spec} \pi_0(A)/(x, x_1, \dots, x_n)$, while the latter is just a point. \square

By induction (and transitivity), one obtains an analogous result for any finite sequence of elements in $\pi_0 A$. Moreover, by replacing A with $A[t_2^{\pm 1}]$, we can thus obtain a result for quotients by even degree elements. We find:

Theorem 2.22. *Let A be a rational noetherian \mathbf{E}_∞ -ring and let $x_1, \dots, x_n \in \pi_{\text{even}}(A)$ be a sequence of elements. Then the map*

$$\pi_{\text{even}}(A)/(x_1, \dots, x_n) \rightarrow \pi_{\text{even}}(A/(x_1, \dots, x_n)),$$

is a finite universal homeomorphism.

3. DEGREE -1 ELEMENTS

We will also encounter odd degree elements in homotopy, and thus, in this section, we consider the free \mathbf{E}_∞ - \mathbb{Q} -algebra $\text{Sym}^*\mathbb{Q}[-1]$ on a generator in degree -1 . It is the purpose of this section to use the coincidence (Proposition 3.2) that $\text{Sym}^*\mathbb{Q}[-1] \simeq C^*(S^1; \mathbb{Q})$ to prove certain basic facts (in Section 3.3) about $\text{Sym}^*\mathbb{Q}[-1] \simeq C^*(S^1; \mathbb{Q})$ -modules, and ultimately about the construction $A//y$ where A is a rational \mathbf{E}_∞ -ring and $y \in \pi_{-1}(A)$.

3.1. The free \mathbf{E}_∞ -ring on $k[-1]$. Let k be a field of characteristic zero, which we will work over. Recall that the free \mathbf{E}_∞ - k -algebra on $k[-1]$ is

$$\text{Sym}^*k[-1] \simeq \bigoplus_{n \geq 0} (k[-1])_{h\Sigma_n}^{\otimes n}.$$

Here $k[-1]^{\otimes n} \simeq k[-n]$, and the Σ_n -action is via the sign representation. For $n \geq 2$, this action is nontrivial, and it follows that the homotopy coinvariants are *zero*. In particular, we find:

Corollary 3.1. *For $\text{char } k = 0$, the homotopy groups of $\text{Sym}^*k[-1]$ are given by:*

$$\pi_i(\text{Sym}^*k[-1]) \simeq \begin{cases} k & \text{if } i = 0 \\ k & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases},$$

and the multiplication is determined (“square zero” in degree -1).

There are two other \mathbf{E}_∞ -rings which have a similar multiplication law on their homotopy groups:

- (1) The cochain \mathbf{E}_∞ -ring on S^1 , $C^*(S^1; k)$.
- (2) The square-zero \mathbf{E}_∞ -ring $k \oplus k[-1]$.

Proposition 3.2. *Let k be a field of characteristic zero. Then there are equivalences of \mathbf{E}_∞ -rings $\text{Sym}^*k[-1] \simeq C^*(S^1; k) \simeq k \oplus k[-1]$.*

Proof. In fact, we can produce maps

$$\text{Sym}^*k[-1] \rightarrow C^*(S^1; k), \quad \text{Sym}^*k[-1] \rightarrow k \oplus k[-1],$$

such that they are isomorphisms on π_{-1} (using the universal property of Sym^*), and therefore are equivalences of \mathbf{E}_∞ -rings by inspection of π_* . So, all three are equivalent. \square

Remark 3.3. If one worked over \mathbb{F}_p , the symmetric algebra $\text{Sym}^*(\mathbb{F}_p[-1])$ is definitely much too large to be either $C^*(S^1; \mathbb{F}_p)$ or $\mathbb{F}_p \oplus \mathbb{F}_p[-1]$, but $C^*(S^1; \mathbb{F}_p)$ and $\mathbb{F}_p \oplus \mathbb{F}_p[-1]$ have the same square-zero multiplication on homotopy groups. They are *not* equivalent as \mathbf{E}_∞ -rings under \mathbb{F}_p because the zeroth reduced power \mathcal{P}^0 acts as the identity on π_{-1} of the former and zero on the latter.

More generally, if n is any *odd* integer, we can repeat the above reasoning:

Corollary 3.4. *If n is odd, then we have equivalences of \mathbf{E}_∞ -rings*

$$(3) \quad \text{Sym}^*k[-n] \simeq k \oplus k[-n],$$

and, if $n > 0$, then these are equivalent to cochains on the n -sphere, $C^(S^n; k)$.*

3.2. Descent properties and $A//x$. If A is a rational \mathbf{E}_∞ -ring and $x \in \pi_*(A)$ is an element in an *even* degree, then we saw in Proposition 2.13 that the construction $A//x$ was reasonably hands-on: it gave us the underlying A -module A/x . If x is in odd degree, however, $A//x$ may be much bigger than A .

Example 3.5. Let $x = 0 \in \pi_{-1}(A)$. Then $A//x \simeq A[t]$.

In fact, for x in an odd degree, it is not even a priori evident that if A is nonzero, then $A//x$ is also nonzero, even as $x^2 = 0$. We will prove this (and more) using descent theory. We recall first a definition.

Definition 3.6. Let $\phi: A \rightarrow A'$ be a morphism of \mathbf{E}_∞ -ring spectra. We say that ϕ *admits descent* if the thick tensor-ideal that A' generates, in $\text{Mod}(A)$, is all of $\text{Mod}(A)$.

We refer to [Mat14a, §3-4] for preliminaries on the notion of “admitting descent.” Here is a simple example.

Proposition 3.7. *Let A be a rational \mathbf{E}_∞ -ring and let $x \in \pi_0 A$ be nilpotent. Then the \mathbf{E}_∞ - A -algebra $A//x$ admits descent over A .*

Proof. In fact, thanks to the octahedral axiom, the thick subcategory of $\text{Mod}(A)$ generated by the A -module A/x (which is equivalent to the underlying A -module of $A//x$) contains $A/x^2, A/x^3, \dots$, and eventually A/x^N where N is so large that $x^N = 0$. But $A/x^N \simeq A \oplus \Sigma A$ for such N , and therefore the thick subcategory generated by $A//x$ actually contains A . \square

Let A be an \mathbf{E}_∞ -ring and let $y \in \pi_n(A)$, with n odd. Then $y^2 = 0$, so that one would hope that the analog of Proposition 3.7 would be automatic. That is, one would hope that $A//y \simeq A \otimes_{\text{Sym}^*(\mathbb{Q}[n])} \mathbb{Q}$ admits descent over A . A priori, it is harder to control this tensor product, because \mathbb{Q} is no longer a perfect $\text{Sym}^*(\mathbb{Q}[n])$ -module for n odd, so one cannot imitate the above argument. However, we can still prove the statement.

Proposition 3.8. *If A is nonzero and $y \in \pi_n A$ (for n odd), then $A//y \in \text{CAlg}_{A/}$ admits descent. In particular, $A//y \neq 0$.*

Proof. The map $\text{Sym}^* \mathbb{Q}[n] \rightarrow \mathbb{Q}$ admits descent, in view of the equivalence $\text{Sym}^*(\mathbb{Q}[n]) \simeq \mathbb{Q} \oplus \mathbb{Q}[n]$ of Corollary 3.4. It follows that the map $A \rightarrow A//y$ admits descent as well by base-change. \square

Example 3.9. Proposition 3.8 definitely fails for \mathbf{E}_∞ -rings under \mathbb{F}_p . For example, if $p = 2$, then odd degree elements can be invertible (take the Tate spectrum $\mathbb{F}_2^{t\mathbb{Z}/2}$). If $p > 2$, odd degree elements square to zero but can still be “resilient.” In the Tate spectrum $\mathbb{F}_p^{t\mathbb{Z}/p}$, we have

$$\pi_*(\mathbb{F}_p^{t\mathbb{Z}/p}) \simeq \mathbb{F}_p[t_2^{\pm 1}] \otimes E(\alpha_{-1}),$$

where the exterior generator α_{-1} has the property that $\beta \mathcal{P}^0 \alpha_{-1} = t_2^{-1}$ is invertible. Thus, even though α_{-1} squares to zero, a basic power operation goes from it to an invertible element. It follows that in any \mathbf{E}_∞ -ring under $\mathbb{F}_p^{t\mathbb{Z}/p}$, if α maps to zero, the whole \mathbf{E}_∞ -ring has to be zero. Such phenomena can never happen in characteristic zero.

3.3. Comparison with local systems. In this subsection, we give the most important (for this paper) application of Proposition 3.2. We will be able to describe the ∞ -category of *modules* over the free algebra $\text{Sym}^* k[-1] \simeq C^*(S^1; k)$ for $\text{char } k = 0$. In fact, let k be any field, not necessarily of characteristic zero. We will describe modules over the cochain algebra $C^*(S^1; k)$. For example, we will be able to give a complete classification of all perfect modules.

We first recall a basic construction from [Mat14a, §7.2].

Definition 3.10. Let X be a space and let \mathcal{C} be an ∞ -category. We define $\text{Loc}_X(\mathcal{C}) = \text{Fun}(X, \mathcal{C})$ and refer to it as the ∞ -category of \mathcal{C} -valued local systems on X .

Construction 3.11. Let A be an \mathbf{E}_∞ -ring and let $\mathcal{C} = \text{Mod}(A)$. Let X be a finite complex. We have a natural fully faithful, symmetric monoidal imbedding

$$\text{Mod}(C^*(X; A)) \subset \text{Loc}_X(\text{Mod}(A)),$$

from modules over the cochain \mathbf{E}_∞ -ring $C^*(X; A)$ into local systems of A -modules on X , whose image in $\mathrm{Loc}_X(\mathrm{Mod}(A))$ is the localizing subcategory generated by the unit.

Informally, the functor sends a $C^*(X; A)$ -module M to the A -module $M \otimes_{C^*(X; A)} A$, which lives as a local system over X (since there is an X 's worth of evaluation maps $C^*(X; A) \rightarrow A$).

In general, it is somewhat subtle to test whether an object in $\mathrm{Loc}_X(\mathrm{Mod}(A))$ belongs to the essential image of $\mathrm{Mod}(C^*(X; A))$. However, if $X = S^1$ then things simplify considerably. Given a local system $N \in \mathrm{Loc}_{S^1}(\mathrm{Mod}(A))$, the evaluation N_q (for a fixed basepoint $q \in S^1$) is an A -module, and N_q acquires a monodromy automorphism ϕ

$$\phi: N_q \simeq N_q,$$

coming from a choice of generator of $\pi_1(S^1; q)$.

Proposition 3.12 ([Mat14a, Remark 7.9]). *Given N as above, then N belongs to the image of $\mathrm{Mod}(C^*(S^1; A))$ if and only if the action of $\phi - 1$ on the homotopy groups $\pi_*(N_q)$ is locally nilpotent.*

It will be important for us to have the correspondence in Proposition 3.12 in as clear terms as possible. Thus, we state the following construction.

Construction 3.13. The right adjoint $\mathrm{Loc}_{S^1}(\mathrm{Mod}(A)) \rightarrow \mathrm{Mod}(C^*(S^1; A))$ is given, for a local system N , by taking its global sections $\varprojlim_{S^1} N$. Explicitly, if $q \in S^1$ and $\phi: N_q \rightarrow N_q$ is as above, we have

$$(4) \quad \varprojlim_{S^1} N = \mathrm{fib} \left(N_q \xrightarrow{\phi - 1} N_q \right),$$

and in particular, we can determine the homotopy groups of $\varprojlim_{S^1} N$ via a long exact sequence.

Remark 3.14. This discussion is special to the case of S^1 . For any finite complex X and any \mathbf{E}_∞ -ring A , we have an inclusion $\mathrm{Mod}(C^*(X; A)) \subset \mathrm{Loc}_X(\mathrm{Mod}(A))$, and the image always is contained in the subcategory of local systems satisfying an ind-unipotence property on homotopy groups, but the precise identification of the image relies on the 1-dimensionality of the circle.

Let \mathcal{C} be an arbitrary ∞ -category. To give a local system on S^1 in some ∞ -category \mathcal{C} is equivalent to giving an object of that ∞ -category and an automorphism, via $S^1 \simeq K(\mathbb{Z}, 1)$. Fix two \mathcal{C} -valued local systems on S^1 , $(x, \phi_x), (y, \phi_y)$, where $x, y \in \mathcal{C}$ and $\phi_x: x \simeq x, \phi_y: y \simeq y$ are automorphisms in \mathcal{C} . Given a map $f: x \rightarrow y$ such that the diagram

$$(5) \quad \begin{array}{ccc} x & \xrightarrow{\phi_x} & x \\ \downarrow f & & \downarrow f \\ y & \xrightarrow{\phi_y} & y \end{array},$$

commutes up to homotopy, then we can produce a map of local systems $(x, \phi_x) \rightarrow (y, \phi_y)$ extending the map $f: x \rightarrow y$.

Remark 3.15. Specifying such a map amounts in addition to *choosing* a homotopy to make the diagram commute, and there may be many homotopy classes of such.

We can state this formally:

Proposition 3.16. *Every object in $\mathrm{Loc}_{S^1}(\mathcal{C})$ is represented by a pair (x, ϕ_x) where $x \in \mathcal{C}$ and $\phi_x: x \rightarrow x$ is an automorphism, and two pairs $(x, \phi_x), (y, \phi_y)$ are isomorphic if and only if there exists an isomorphism $f: x \rightarrow y$ such that the diagram (5) is homotopy commutative.*

We now specialize to the case where $A = k$ is a field. We will use Construction 3.11, Proposition 3.12, and Proposition 3.16 to classify $C^*(S^1; k)$ -modules.

Proposition 3.17. *Any local system $\mathcal{L} \in \mathrm{Loc}_{S^1}(\mathrm{Mod}(k))$ decomposes uniquely as a direct sum $\mathcal{L} \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n[n]$ where the fiber of \mathcal{L}_n at a point of S^1 is discrete.*

Proof. Let $\mathcal{L} = (M, \phi)$. The k -module M decomposes as a sum of its homotopy groups, i.e., $M \simeq \bigoplus_{n \in \mathbb{Z}} (\pi_n M)[n]$, and the automorphism $\phi: M \rightarrow M$ is determined by its behavior on its homotopy groups. It follows that, for each n , we can produce squares

$$\begin{array}{ccc} (\pi_n M)[n] & \xrightarrow{\phi_*} & (\pi_n M)[n] \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & M \end{array}$$

which commute up to homotopy. Putting this together, we can produce a square

$$\begin{array}{ccc} \bigoplus_{n \in \mathbb{Z}} (\pi_n M)[n] & \xrightarrow{\phi_*} & \bigoplus_{n \in \mathbb{Z}} (\pi_n M)[n] \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & M \end{array}$$

which commutes up to homotopy, and where the vertical maps are now equivalences. It follows from Proposition 3.16 that the pair (M, ϕ) is equivalent to the direct sum of the pairs $\{((\pi_n M)[n], \phi_*)\}_{n \in \mathbb{Z}}$, where each of these is concentrated in a single degree. \square

Using the imbedding $\text{Mod}(C^*(S^1; k)) \subset \text{Loc}_{S^1}(\text{Mod}(k))$, we find from this:

Corollary 3.18. *Any $C^*(S^1; k)$ -module M admits a unique decomposition $M \simeq \bigoplus_{n \geq 0} M_n[n]$, where $M_n \in \text{Mod}(C^*(S^1; k))$ has the property that $M \otimes_{C^*(S^1; k)} k$ is a discrete k -module.*

In order to give a discrete local system on S^1 , it suffices simply to give a k -vector space with an automorphism. In the case we are interested, i.e., local systems coming from $C^*(S^1; k)$ -modules, it follows that to give an equivalence class of $C^*(S^1; k)$ -modules M equates to giving, for each $n \in \mathbb{Z}$, a discrete $k[x]$ -module on which x acts *locally nilpotently*. The n th such object corresponds to $\pi_n(M \otimes_{C^*(S^1; k)} k)$ and x is the monodromy automorphism minus the identity.

In general, the classification of torsion modules over a PID is nontrivial, but in the finitely generated case, we have a simple complete classification. This leads to:

Construction 3.19. Fix $i \in \mathbb{Z}_{>0}$. We consider the n -dimensional k -vector space $V_i = k[x]/x^n$ and the nilpotent endomorphism given by multiplication by x . Let \mathcal{V}_i be the associated local system on S^1 with fiber V_i and monodromy automorphism $1 + x$. Then, by Proposition 3.12, \mathcal{V}_i corresponds to a $C^*(S^1; k)$ -module that we will denote by N_i . Thus, $N_i \otimes_{C^*(S^1; k)} k$ is discrete, and i -dimensional, and the monodromy automorphism is unipotent with a single Jordan block.

In order to determine the homotopy groups of N_i , we have to form the associated local system, and take global sections over S^1 , as in Construction 3.13. We have

$$\pi_j(N_i) = \begin{cases} k & \text{if } j = 0 \\ k & \text{if } j = -1 \\ 0 & \text{otherwise} \end{cases}.$$

For example, $N_1 = C^*(S^1; k)$.

Proposition 3.20. *N_i is a perfect $C^*(S^1; k)$ -module.*

Proof. By construction, $N_i \otimes_{C^*(S^1; k)} k$ is a perfect k -module. Moreover, $C^*(S^1; k) \rightarrow k$ admits descent [Mat14a, Prop. 3.35]. Therefore, N_i is a perfect $C^*(S^1; k)$ -module in view of [Mat14a, Prop. 3.27]. \square

Proposition 3.21. *Any perfect $C^*(S^1; k)$ -module M decomposes uniquely as a sum of copies of shifts of the N_i .*

Proof. Given a perfect $C^*(S^1; k)$ -module, the associated local system (which is a local system of perfect k -modules) splits as a direct sum of shifts of local systems of discrete k -modules, by Proposition 3.17. Each of these is determined by a finite-dimensional k -vector space with an unipotent automorphism, and these are classified as a direct sum of indecomposable ones determined by their

Jordan type, corresponding to the decomposition of a finitely generated $k[x]$ -module as a direct sum of cyclic ones. These summands correspond to the $C^*(S^1; k)$ -modules N_i . \square

3.4. Evenly graded $C^*(S^1; k)$ -modules. Let k be a field. We will have to work with certain non-perfect $C^*(S^1; k)$ -modules in the sequel, and here we will prove a basic technical result (Proposition 3.23) concerning them. We need the following classical algebraic fact with $R = k[x]_{(x)}$.

Theorem 3.22. *Let R be a discrete valuation ring with quotient field K . Then any torsion divisible R -module is a direct sum of copies of K/R .*

We refer to [KT08, Theorem 6.3] for a proof of Theorem 3.22. We will translate it into our setting and prove the following.

Proposition 3.23. *Let M be a $C^*(S^1; k)$ -module. Suppose that $\pi_i M = 0$ if i is odd. Then M is a direct sum of copies of even shifts of the $C^*(S^1; k)$ -module k (under the map of \mathbf{E}_∞ -rings $C^*(S^1; k) \rightarrow k$ given by evaluation at a point).*

Proof. We know, first, that M is a direct sum of copies of $C^*(S^1; k)$ -modules whose base-change to k is shifted discrete (Proposition 3.17), so we may assume this to begin with. That is, we may assume that $M \otimes_{C^*(S^1; k)} k$ is concentrated in one degree, say n , in homotopy. In particular, under the correspondence between $C^*(S^1; k)$ -modules and local systems with the unipotence property, M comes from a discrete $k[x]$ -module P_0 , on which x is locally nilpotent. Moreover, at most one of $\ker x$, $\operatorname{coker} x$ can be nonzero because of the hypothesis on the homotopy groups on M , and Construction 3.13, which describes how to get from P_0 to M .

Since x is locally nilpotent, $\ker x$ is always nonzero (if $P_0 \neq 0$), so the conclusion must be that $\operatorname{coker} x = 0$, and we have an even shift of a discrete local system. In other words, P_0 is a x -torsion divisible $k[x]$ -module. Any such is a direct sum of copies of $k[x^{\pm 1}]/k[x]$ by Theorem 3.22.

It follows that if \widetilde{M} is the $C^*(S^1; k)$ -module corresponding to the $k[x]$ -module $k[x^{\pm 1}]/k$, then M is a direct sum of even shifts of copies of \widetilde{M} . It remains to argue that $k \simeq \widetilde{M}$. In fact, our reasoning shows that k must be a direct sum of copies of \widetilde{M} , but clearly k is indecomposable, so $k \simeq \widetilde{M}$. \square

Corollary 3.24. *Let M be a $C^*(S^1; k)$ -module such that $\pi_i M = 0$ if i is odd. Then the k -module $M \otimes_{C^*(S^1; k)} k$ has the same property.*

Proof. This follows from Proposition 3.23, but it could also have been seen directly. \square

4. RESIDUE FIELDS

In this section, we will prove Theorems 1.2 and 1.3 on the existence of residue fields and the detection of nilpotence. It will be convenient to work throughout with an extra assumption of a degree two unit. In this case, the attachment of even cells can always be replaced with the attachment of degree zero cells.

4.1. Definitions. Let A be a rational, noetherian \mathbf{E}_∞ -ring such that $\pi_2(A)$ contains a unit. Fix a prime ideal $\mathfrak{p} \subset \pi_0(A)$.

Definition 4.1. A *residue field* for A is an object $\kappa(\mathfrak{p}) \in \mathbf{CAlg}_{A/}$ such that:

- (1) The map $\pi_0(A) \rightarrow \pi_0(\kappa(\mathfrak{p}))$ exhibits $\pi_0(\kappa(\mathfrak{p}))$ as the residue field of $\pi_0(A)$ at \mathfrak{p} .
- (2) $\pi_1(\kappa(\mathfrak{p})) = 0$.

In particular, if $k(\mathfrak{p})$ is the residue field of $\pi_0 A$ at \mathfrak{p} , then $\pi_*(\kappa(\mathfrak{p}))$ is a Laurent series ring on $k(\mathfrak{p})$ on a generator in degree two.

In this section, we will show that residue fields for such \mathbf{E}_∞ -rings exist uniquely, and are sufficient to detect nilpotence in $\operatorname{Mod}(A)$. The rest of the paper will use these residue fields to describe certain invariants of $\operatorname{Mod}(A)$.

Remark 4.2. The name “residue field” is appropriate because of the perfect Künneth isomorphism

$$\kappa(\mathfrak{p})_*(M) \otimes_{\kappa(\mathfrak{p})_*} \kappa(\mathfrak{p})_*(N) \simeq \kappa(\mathfrak{p})_*(M \otimes_A N), \quad M, N \in \text{Mod}(A);$$

indeed, there is a map from left to right which is an isomorphism for $M = N = A$, and both sides define two-variable homology theories on $\text{Mod}(A)$, so the natural map must be an isomorphism in general. Alternatively, any $\kappa(\mathfrak{p})$ -module is a sum of shifts of free ones.

We describe the connection with the use of residue fields as in [BR08]. Given an even periodic \mathbf{E}_∞ -ring A (not necessarily over \mathbb{Q}) with $\pi_0(A)$ *regular* noetherian, it is possible to form “residue fields” of A as \mathbf{E}_1 -algebras in $\text{Mod}(R)$, by successively quotienting by a regular sequence. These residue fields have analogous properties of detecting nilpotence [Mat14b, Cor. 2.6] and are quite useful for describing invariants of $\text{Mod}(A)$ (e.g., [BR05, BR08, Mat14a]). These residue fields are usually *not* \mathbf{E}_∞ -algebras in $\text{Mod}(A)$. For example, in the “chromatic” setting, the associated residue fields (such as the Morava K -theories $K(n)$ for the \mathbf{E}_∞ -ring E_n) are almost *never* \mathbf{E}_∞ .

Over the rational numbers, we are able to produce residue fields without such regularity hypotheses, and as \mathbf{E}_∞ -algebras. However, we will have to work a bit harder: the residue fields of such an A will no longer in general be perfect as A -modules (or as \mathbf{E}_∞ - A -algebras), and we will have to use a countable limiting procedure, together with the techniques from the previous sections.

Let A be as above. In order to construct a residue field for A for the prime ideal $\mathfrak{p} \in \text{Spec} \pi_0 A$, we may first localize at \mathfrak{p} , and assume that $\pi_0 A$ is *local* and that \mathfrak{p} is the *maximal ideal*. Then, given generators $x_1, \dots, x_n \in \pi_0 A$ for \mathfrak{p} , we will need to set them equal to zero by attaching 1-cells. That of course will introduce new elements (in both π_0, π_1) and we will have to kill them in turn. This process will be greatly facilitated by the analysis in Section 3.

4.2. Detection of nilpotence. Given an \mathbf{E}_∞ -ring A , we start by reviewing what it means for a collection of A -algebras to *detect nilpotence*, following ideas of [DHS88, HS98].

Definition 4.3. Let A be an \mathbf{E}_∞ -ring, and let A' be an A -ring spectrum: that is, an associative algebra object in the *homotopy category* of $\text{Mod}(A)$. We say that $A \rightarrow A'$ *detects nilpotence* if, whenever T is an A -ring spectrum, then the map of associative rings

$$\pi_*(T) \rightarrow \pi_*(A' \otimes_A T)$$

has the property that any $u \in \pi_*(T)$ which maps to a nilpotent element is nilpotent. More generally, a collection of A -ring spectra $\{A'_\alpha\}_{\alpha \in S}$ is said to *detect nilpotence* if any $u \in \pi_*(T)$ which maps to nilpotent elements under each map $\pi_*(T) \rightarrow \pi_*(A'_\alpha \otimes_A T)$ is itself nilpotent.

For example, the nilpotence theorem (Theorem 1.1) states that the Morava K -theories and homology (rational and mod p) detect nilpotence for $A = S^0$. The original form (in [DHS88]) states that the \mathbf{E}_∞ -ring MU of complex bordism detects nilpotence by itself, again over S^0 .

As in [DHS88, §1], one has the following consequences of detecting nilpotence:

Proposition 4.4. *Let $\{A'_\alpha\}_{\alpha \in S}$ be a collection of A -ring spectra that detect nilpotence.*

- (1) *Given a map of perfect A -modules $\phi: T \rightarrow T'$ such that each $1_{A'_\alpha} \otimes_A \phi: A'_\alpha \otimes_A T \rightarrow A'_\alpha \otimes_A T'$ is nullhomotopic as a map of A -modules, then ϕ is smash nilpotent: $\phi^{\otimes N}: T^{\otimes N} \rightarrow T'^{\otimes N}$ is nullhomotopic for $N \gg 0$.*
- (2) *Given a self-map of perfect A -modules $v: \Sigma^k T \rightarrow T$, if each $1_{A'_\alpha} \otimes_A v: \Sigma^k(A'_\alpha \otimes_A T) \rightarrow A'_\alpha \otimes_A T$ is nilpotent in $\text{Mod}(A)$, then v itself is nilpotent.*

Example 4.5. Suppose A' is an A -ring spectrum that detects nilpotence. Then A' cannot annihilate any nonzero perfect A -module M ; in fact, that would force the identity $M \rightarrow M$ to be nilpotent.

Moreover, one sees:

Proposition 4.6. *Let A be an \mathbf{E}_∞ -ring.*

- (1) *Let $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$ be a diagram of A -ring spectra, such that the colimit $A_\infty = \varinjlim A_i$ has a compatible structure of an A -ring spectrum. If each A_i detects nilpotence over A , then A_∞ detects nilpotence over A .*

- (2) Let $\{A'_\alpha\}_{\alpha \in S}$ be a collection of \mathbf{E}_∞ - A -algebras that detect nilpotence over A . For each $\alpha \in S$, let $\{A''_{\alpha\beta}\}_{\beta \in T_\alpha}$ be a collection of \mathbf{E}_∞ - A'_α -algebras that detect nilpotence over A'_α . Then the collection $\{A''_{\alpha\beta}\}_{\alpha \in S, \beta \in T_\alpha}$ of A -algebras detects nilpotence over A .
- (3) Suppose the $\{A'_\alpha\}_{\alpha \in S}$ are a collection of \mathbf{E}_∞ - A -algebras detecting nilpotence over A such that each $\pi_*(A'_\alpha)$ is a graded field. Then an A -ring spectrum A'' detects nilpotence over R if and only if $A'' \otimes_A A'_\alpha \neq 0$ for each $\alpha \in S$.

Proof. By a *graded field*, we mean a graded ring which is either a field (concentrated in degree zero) or $k[t^{\pm 1}]$ for $|t| > 0$ and k a field. The third assertion then follows from the second, since any nonzero ring spectrum over each A'_α detects nilpotence over A'_α . The proofs of the first and second assertions are straightforward. \square

Finally, we need an important example of a pair that detects nilpotence.

Example 4.7. Let A be a rational \mathbf{E}_∞ -ring, and let $x \in \pi_0(A)$. As before, the cofiber A/x inherits the canonical structure of an \mathbf{E}_∞ -algebra under A , as $A//x$. The localization $A[x^{-1}]$ always inherits a natural \mathbf{E}_∞ -ring structure. The claim is that the pair of A -algebras $\{A/x, A[x^{-1}]\}$ detects nilpotence.

To see this, let T be an A -ring spectrum, and let $\alpha \in \pi_j(A)$. Suppose α maps to zero in $A[x^{-1}] = T \otimes_A A[x^{-1}]$. This means that $x^N \alpha = 0$ for N chosen large enough. Suppose also that α maps to zero in $\pi_j(T/x) = \pi_j(T \otimes_A A/x)$. This means that $\alpha = x\beta$ for some $\beta \in \pi_j(T)$. We then have

$$\alpha^{2N} = \alpha^N \alpha^N = (x\beta)^N \alpha^N = \beta^N x^N \alpha^N = 0,$$

since $x^N \alpha = 0$. In other words, α is nilpotent.

This example will be extremely important to us in making induction arguments on the Krull dimension.

Example 4.8. Let $A \rightarrow A'$ be a map of \mathbf{E}_∞ -rings. Suppose that $A \rightarrow A'$ admits descent (Definition 3.6). Then $A \rightarrow A'$ detects nilpotence; see for instance [Mat14a, Prop. 3.26].

4.3. The main result. In this subsection, we prove the main technical result of this paper (Theorem 4.14): the existence of residue fields and the detection of nilpotence. We begin with a preliminary technical result.

Proposition 4.9. *Let B be a rational \mathbf{E}_∞ -ring such that:*

- (1) $\pi_0(B)$ is a field k .
- (2) $\pi_2(B)$ contains a unit u .
- (3) $\pi_{-1}(B)$ is a countably dimensional k -vector space.

Then there exists a sequence of \mathbf{E}_∞ -rings

$$(6) \quad B = B^{(0)} \rightarrow B^{(1)} \rightarrow B^{(2)} \rightarrow \dots$$

such that:

- (1) Each $B^{(i)}$ satisfies the three hypotheses above on B .
- (2) There exists an element $y_i \in \pi_{-1}(B^{(i-1)})$ such that $B^{(i)} \simeq B^{(i-1)}/y_i$.
- (3) Given any element $y \in \pi_{-1}(B)$, there exists N such that y maps to zero under the map $B \rightarrow B^{(N)}$.

Proof. Note first that, by Proposition 2.15, B naturally admits the structure of an \mathbf{E}_∞ - k -algebra. Let $u_1, u_2, \dots \in \pi_{-1}(B)$ be a k -basis. We define $B^{(1)} \simeq B//u_1 = B \otimes_{\text{Sym}^* k[-1]} k$ via the map $\text{Sym}^* k[-1] \rightarrow B$ classifying u_1 . Next, we define the \mathbf{E}_∞ - $B^{(1)}$ -algebra $B^{(2)} \simeq B^{(1)}/u_2 \simeq B^{(1)} \otimes_{\text{Sym}^* k[-1]} k$ where the map $\text{Sym}^* k[-1] \rightarrow B^{(1)}$ classifies u_2 . Inductively, we obtain a sequence of \mathbf{E}_∞ -rings $B^{(i)}$. We need to verify the above three conclusions on the $B^{(i)}$. The second and third conclusions are immediate from the construction.

Consider the $\text{Sym}^* k[-1]$ -module B under the map $\text{Sym}^* k[-1] \rightarrow B$ classifying u_1 . We have a map

$$k[t_2^{\pm 1}] \otimes_k \text{Sym}^* k[-1] \rightarrow B,$$

of $\mathrm{Sym}^*k[-1]$ -modules. The cofiber C , by hypothesis, is a $\mathrm{Sym}^*k[-1]$ -module whose homotopy groups are concentrated *entirely* in odd degrees. In particular, we find by Proposition 3.23 that C is a direct sum of *odd* shifts of the $\mathrm{Sym}^*k[-1]$ -module k , so the cofiber sequence

$$k[t_2^{\pm 1}] \rightarrow B \otimes_{\mathrm{Sym}^*k[-1]} k \rightarrow C \otimes_{\mathrm{Sym}^*k[-1]} k,$$

(which has to induce *split* exact sequences on the level of homotopy groups) implies that, at the level of homotopy groups, $B^{(1)} = B \otimes_{\mathrm{Sym}^*k[-1]} k$ has the same property as did B : the even homotopy groups are given by the Laurent series ring. Moreover, the map $\pi_*(C) \rightarrow \pi_*(C \otimes_{\mathrm{Sym}^*k[-1]} k)$ is injective, so the $\{u_2, u_3, \dots\}$ remain nonzero and linearly independent in $\pi_{-1}(B^{(1)})$. We find inductively that all the $B^{(i)}$ have homotopy groups entirely in odd degrees except for the Laurent series over k . □

Lemma 4.10. *Let k be a field of characteristic zero and let A be any \mathbf{E}_∞ -ring. Then the map*

$$(7) \quad \mathrm{Hom}_{\mathrm{CAlg}}(k[t_2^{\pm 1}], A) \rightarrow \mathrm{Hom}_{\mathrm{Ring}_*}(\pi_*(k[t_2^{\pm 1}]), \pi_*(A))$$

is a bijection.

Proof. This follows as in the proof Proposition 2.15. That is, using a transcendence basis for k over \mathbb{Q} , one sees that there exists a free \mathbf{E}_∞ -ring on a discrete \mathbb{Q} -module V such that k is a filtered colimit of étale $\mathrm{Sym}^*(V)$ -algebras. The analog of (7) is easily seen to be true for $\mathrm{Sym}^*(V)$ and, by the theory of étale extensions [Lur14, §7.5], it must hold for k . In other words, the map

$$(8) \quad \mathrm{Hom}_{\mathrm{CAlg}}(k, A) \rightarrow \mathrm{Hom}_{\mathrm{Ring}_*}(k, \pi_0(A))$$

is a bijection. It is similarly easy to see that the analog holds for k replaced by $\mathbb{Q}[t_2^{\pm 1}]$. Since $k[t_2^{\pm 1}] \simeq k \otimes_{\mathbb{Q}} \mathbb{Q}[t_2^{\pm 1}]$, we may conclude. □

Proposition 4.11. *Let B satisfy the hypotheses of Proposition 4.9.*

(1) *Then there exists a map of \mathbf{E}_∞ -rings $B \rightarrow k[t_2^{\pm 1}]$ which detects nilpotence.*

(2) *If L is any field of characteristic zero, then the map*

$$(9) \quad \pi_0 \mathrm{Hom}_{\mathrm{CAlg}}(B, L[t_2^{\pm 1}]) \rightarrow \mathrm{Hom}_{\mathrm{Ring}_*}(\pi_*(B), \pi_*(L[t_2^{\pm 1}]))$$

is a bijection.

Proof. Given B and the basis u_1, u_2, \dots , as above, we let B_1 be the colimit $\varinjlim B^{(i)}$ of the sequence (6) obtained by applying Proposition 4.9. Then B_1 satisfies the hypotheses of this proposition as well. Moreover, the map $\pi_{-1}(B) \rightarrow \pi_{-1}(B_1)$ is the zero map. Thus, we can repeat the above sequential construction of Proposition 4.9 to B_1 to produce a new sequence

$$B_1 \rightarrow B_1^{(1)} \rightarrow B_1^{(2)} \rightarrow \dots,$$

obtained by iteratively coning off the degree -1 elements in $\pi_*(B_1)$. Let the colimit be the \mathbf{E}_∞ - B_1 -algebra B_2 . Then B_2 satisfies the hypotheses of Proposition 4.9, but $\pi_{-1}(B_1) \rightarrow \pi_{-1}(B_2)$ is zero. Repeating the process, we get a sequence

$$(10) \quad B \rightarrow B_1 \rightarrow B_2 \rightarrow \dots,$$

whose colimit, finally, is the \mathbf{E}_∞ -ring $k[t_2^{\pm 1}]$, since each of the maps is zero on π_{-1} .

We need to see that the map $B \rightarrow k[t_2^{\pm 1}]$ thus constructed detects nilpotence. For this, it suffices to argue that each $B_i \rightarrow B_{i+1}$ in the above sequence detects nilpotence, since detecting nilpotence is preserved in filtered colimits. But $B_i \rightarrow B_{i+1}$ is a filtered colimit of maps each of which is obtained by coning off a degree -1 element, and these maps detect nilpotence by Proposition 3.8.

Finally, we need to analyze homotopy classes of maps $B \rightarrow L[t_2^{\pm 1}]$ where L is a field of characteristic zero. We have already shown that (9) is a surjection, so we only need to prove injectivity. For this, we observe that any map $B \rightarrow L[t_2^{\pm 1}]$ extends over $B \rightarrow B_1$. Indeed, $B \rightarrow B_1$ is a filtered colimit of maps $B^{(i-1)} \rightarrow B^{(i-1)}/y_i$ where $|y_i| = -1$. Since $\pi_{-1}L[t_2^{\pm 1}] = 0$, any map $B^{(i-1)} \rightarrow L[t_2^{\pm 1}]$ can be extended over $B^{(i)}$. In particular, $\pi_0 \mathrm{Hom}_{\mathrm{CAlg}}(B^{(i)}, L[t_2^{\pm 1}]) \rightarrow \pi_0 \mathrm{Hom}_{\mathrm{CAlg}}(B^{(i-1)}, L[t_2^{\pm 1}])$ is a surjection. Taking inverse limits, it follows easily that $\pi_0 \mathrm{Hom}_{\mathrm{CAlg}}(B_1, L[t_2^{\pm 1}]) \rightarrow \pi_0 \mathrm{Hom}_{\mathrm{CAlg}}(B, L[t_2^{\pm 1}])$

is a surjection too, as claimed. Applying this to B_i , we find that each map $\pi_0 \text{Hom}_{\text{CAlg}}(B_i, L[t_2^{\pm 1}]) \rightarrow \pi_0 \text{Hom}_{\text{CAlg}}(B_{i-1}, L[t_2^{\pm 1}])$, is a surjection, and taking limits, that the map

$$\pi_0 \text{Hom}_{\text{CAlg}}(k[t_2^{\pm 1}], L[t_2^{\pm 1}]) \rightarrow \pi_0 \text{Hom}_{\text{CAlg}}(B, L[t_2^{\pm 1}])$$

is a surjection, too. However, maps $k[t_2^{\pm 1}] \rightarrow L[t_2^{\pm 1}]$ of \mathbf{E}_∞ -rings are determined by their action on homotopy groups in view of Lemma 4.10. This proves uniqueness and completes the proof of Proposition 4.11. \square

Proposition 4.12. *Let A_0 be a noetherian, rational \mathbf{E}_∞ -ring containing a unit in degree two. Suppose $\pi_0(A_0)$ is a local artinian ring with residue field k . Let L be any field of characteristic zero. Then:*

- (1) *The natural map $\pi_0 \text{Hom}_{\text{CAlg}}(A_0, L[t_2^{\pm 1}]) \rightarrow \text{Hom}_{\text{Ring}_*}(\pi_* A_0, L[t_2^{\pm 1}]) \simeq \text{Hom}_{\text{Ring}_*}(k[t_2^{\pm 1}], L[t_2^{\pm 1}])$ is a bijection.*
- (2) *Any map $A_0 \rightarrow L[t_2^{\pm 1}]$ of \mathbf{E}_∞ -rings detects nilpotence.*

Proof. We first treat existence in case $L = k$. Our goal is to produce a map of \mathbf{E}_∞ -rings from A_0 to $k[t_2^{\pm 1}]$. For this, we will need to kill the degree zero elements, and the degree -1 elements. We will first kill the degree 0 elements by adding cells in dimension one, to reduce to the case where there is nothing (except for k) in odd dimensions. Then, we will use a separate argument to kill the odd homotopy.

We first make A_0 into an \mathbf{E}_∞ - k -algebra, using Proposition 2.15. Given A_0 , let $\mathfrak{m} \subset \pi_0(A_0)$ be the maximal ideal, which is a finite-dimensional k -vector space. Consider the map

$$\text{Sym}^*(\mathfrak{m}) \rightarrow A_0,$$

of \mathbf{E}_∞ - k -algebras, and form the pushout

$$A_1 \stackrel{\text{def}}{=} A_0 \otimes_{\text{Sym}^*(\mathfrak{m})} k,$$

which has the same property as A_0 : $\pi_*(A_1)$ satisfies the desired noetherianness assumptions (in fact, all the homotopy groups are finite-dimensional k -vector spaces), and $\pi_0(A_1)$ is local artinian with residue field k by Theorem 2.22. Note that $A_0 \rightarrow A_1$ admits descent in view of Proposition 3.7.

Let $\mathfrak{m}_1 \subset \pi_0(A_1)$ be the maximal ideal, and continue the process with

$$A_2 \stackrel{\text{def}}{=} A_1 \otimes_{\text{Sym}^*(\mathfrak{m}_1)} k,$$

and repeating this, we find a sequence of 2-periodic, noetherian \mathbf{E}_∞ -rings

$$A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots,$$

where each A_i has the following properties:

- (1) $\pi_0(A_i)$ is a local artinian ring with residue field k and $\pi_1(A_i)$ is a finite-dimensional k -vector space.
- (2) A_{i+1} is obtained from A_i by attaching a 1-cell for each element in a k -basis of the maximal ideal of $\pi_0(A_i)$, and thus $A_i \rightarrow A_{i+1}$ admits descent.
- (3) In particular, the map $\pi_0(A_i) \rightarrow \pi_0(A_{i+1})$ annihilates the maximal ideal of the former.

If we take the colimit $A_\infty = \varinjlim_i A_i$, we find an \mathbf{E}_∞ - A -algebra A_∞ such that $\pi_0(A_\infty) = k$. This process of iteratively attaching 1-cells has likely introduced elements in π_{-1} , but we know that $\pi_{-1}(A_\infty)$ is a *countably* dimensional k -vector space. Note that $A_0 \rightarrow A_\infty$ detects nilpotence, as it is the sequential colimit of a sequence of objects in $\text{CAlg}_{A_0/}$ that admit descent. But now we can appeal to Proposition 4.11 to obtain a map $A_\infty \rightarrow k[t_2^{\pm 1}]$ which detects nilpotence. This completes the proof of existence.

We may handle the first assertion (of which we now only need to prove *injectivity* of the map) in a similar manner as in the proof of Proposition 4.11. That is, we observe that

$$\text{Hom}_{\text{CAlg}}(A_i, L[t_2^{\pm 1}]) \rightarrow \text{Hom}_{\text{CAlg}}(A_{i-1}, L[t_2^{\pm 1}])$$

is surjective for each i , because $A_i \simeq A_{i-1}/(x_i^{(1)}, \dots, x_i^{(n)})$ where the $x_i^{(j)}$ are nilpotent. In the limit, we find that the map

$$\text{Hom}_{\text{CAlg}}(A_\infty, L[t_2^{\pm 1}]) \rightarrow \text{Hom}_{\text{CAlg}}(A_0, L[t_2^{\pm 1}])$$

is surjective. But now we can apply the uniqueness statement of Proposition 4.11 to complete the proof. That is, maps $A_\infty \rightarrow L[t_2^{\pm 1}]$ are determined by their behavior on homotopy groups. So, if we had two different maps $A_0 \rightarrow L[t_2^{\pm 1}]$ inducing the same behavior on homotopy groups, we would get two different maps $A_\infty \rightarrow L[t_2^{\pm 1}]$ inducing the same behavior on homotopy groups, and this is a contradiction. \square

Proposition 4.13. *Let A be a noetherian, rational \mathbf{E}_∞ -ring containing a unit in degree two. Suppose k is a field of characteristic zero. Then the map*

$$\pi_0 \mathrm{Hom}_{\mathrm{CAlg}}(A, k[t_2^{\pm 1}]) \rightarrow \mathrm{Hom}_{\mathrm{Ring}_*}(\pi_*(A), k[t_2^{\pm 1}])$$

is a bijection.

Proof. We begin with surjectivity. Suppose given a map $\pi_0(A) \rightarrow k$. We want to realize this at the level of \mathbf{E}_∞ -rings. By localizing, we may assume that $\pi_0(A)$ is a local ring and that $\mathfrak{p} \subset \pi_0(A)$ is its maximal ideal; let k' be the residue field $\pi_0(A)/\mathfrak{p}$. By hypothesis, we have a map $k' \rightarrow k$. Choose ideal generators $x_1, \dots, x_n \in \mathfrak{p}$ and use them to construct a map

$$\mathbb{Q}[u_1, \dots, u_n] \rightarrow A, \quad u_i \mapsto x_i.$$

We let A_0 be the \mathbf{E}_∞ -ring $A \otimes_{\mathbb{Q}[u_1, \dots, u_n]} \mathbb{Q} \simeq A/(u_1, \dots, u_n)$. Then we get a map $A \rightarrow A_0 \rightarrow k'[t_2^{\pm 1}]$ by Proposition 4.12 realizing the map on π_* desired. We can compose this with the map $k[t_2^{\pm 1}] \rightarrow k'[t_2^{\pm 1}]$.

Suppose now we have two distinct \mathbf{E}_∞ -maps $A \rightarrow k[t_2^{\pm 1}]$ realizing the same map on π_* . The same argument as before shows that both maps extend to (necessarily distinct) maps $A_0 \rightarrow k[t_2^{\pm 1}]$ realizing the same map on homotopy groups, but this contradicts Proposition 4.12. \square

We can now prove our main result.

Theorem 4.14. *Let A be a noetherian, rational \mathbf{E}_∞ -ring containing a unit in degree two.*

- (1) *For each prime ideal $\mathfrak{p} \subset \pi_0(A)$, a residue field $\kappa(\mathfrak{p}) \in \mathrm{CAlg}_{A/}$ for A at \mathfrak{p} exists and is unique up to homotopy.*
- (2) *The \mathbf{E}_∞ - A -algebras $\kappa(\mathfrak{p})$ detect nilpotence.*

Proof. The existence and uniqueness of residue fields is a consequence of Proposition 4.13, since we can of course construct them uniquely in the setting of commutative algebra (i.e., at the level of homotopy groups).

Finally, we need to show that the residue fields detect nilpotence. If $\pi_0 A$ is local artinian, then we have already seen this (Proposition 4.12). Now, assume the result on detection of nilpotence proved for all noetherian A with the Krull dimension of $\pi_0(A)$ at most $n - 1$. We will then prove it for dimension $\leq n$. In fact, we may assume that $\pi_0 A$ is noetherian *local* of Krull dimension $\leq n$. Choose $x \in \pi_0 A$ such that $\pi_0 A/(x)$ has Krull dimension $\leq n - 1$. As we saw in Example 4.7, the pair of \mathbf{E}_∞ - A -algebras $A/x, A[x^{-1}]$ detect nilpotence over A , and each of these is noetherian with π_0 having Krull dimension $\leq n - 1$. Therefore, the residue fields of the \mathbf{E}_∞ -rings $A/x, A[x^{-1}]$ are sufficient to detect nilpotence over each of them, and thus detect nilpotence over A by Proposition 4.6.

Given *any* rational noetherian \mathbf{E}_∞ -ring A , in order to prove that the residue fields $\{\kappa(\mathfrak{p})\}_{\mathfrak{p} \in \mathrm{Spec} \pi_0 A}$ detect nilpotence over A , it suffices to reduce to the case where $\pi_0 A$ is *local*, and thus of finite Krull dimension, so that what we have already done suffices to show that the residue fields detect nilpotence. \square

Proposition 4.15. *Let A be a rational noetherian \mathbf{E}_∞ -ring containing a unit in π_2 . Suppose $B \in \mathrm{CAlg}_{A/}$ is noetherian as well. Let $\mathfrak{p} \in \mathrm{Spec} \pi_0 A$. Let $k(\mathfrak{p})$ be the associated algebraic residue field and let $\kappa(\mathfrak{p}) \in \mathrm{CAlg}_{A/}$ be the topological one. Then the following are equivalent:*

- (1) $\pi_0(B) \otimes_{\pi_0(A)} k(\mathfrak{p}) \neq 0$.
- (2) $B \otimes_A \kappa(\mathfrak{p}) \neq 0$.

Proof. There is a map of commutative rings $\pi_0(B) \otimes_{\pi_0(A)} k(\mathfrak{p}) \rightarrow \pi_0(B \otimes_A \kappa(\mathfrak{p}))$, so if the latter is nonzero, clearly the former is as well. Suppose the former is nonzero now. Without loss of generality, we may assume that $\pi_0(A)$ is local and \mathfrak{p} is its maximal ideal. Let (x_1, \dots, x_n) be a system of generators for \mathfrak{p} and let $A_0 = A/(x_1, \dots, x_n)$. Then $B \otimes_A A_0 \neq 0$ as $\pi_0(B)/(x_1, \dots, x_n) \neq 0$, in view of Theorem 2.22; it is here that we use that B is noetherian. However, the map $A_0 \rightarrow \kappa(\mathfrak{p})$ detects nilpotence as $\pi_0(A_0)$ is local artinian, so that if $B \otimes_A A_0 \neq 0$, then $B \otimes_A \kappa(\mathfrak{p}) \neq 0$ too. \square

Remark 4.16. Proposition 4.15 is false if we do not assume B is noetherian. We refer to Section 8.2 for a counterexample: the map $\mathbb{Q}[x, y] \rightarrow A$ constructed is an isomorphism on π_0 , but $B/(x, y) = 0$.

Corollary 4.17. *Let A be a rational noetherian \mathbf{E}_∞ -ring containing a unit in π_2 . Given two different prime ideals $\mathfrak{p}, \mathfrak{q} \subset \pi_0 A$, the tensor product $\kappa(\mathfrak{p}) \otimes_A \kappa(\mathfrak{q})$ is contractible.*

Corollary 4.18. *Let A be a rational noetherian \mathbf{E}_∞ -ring containing a unit in degree two. Let $A', A'' \in \mathrm{CAlg}_{A/}$ be noetherian. Then the following are equivalent:*

- (1) $\pi_0(A') \otimes_{\pi_0(A)} \pi_0(A'') \neq 0$.
- (2) $A' \otimes_A A'' \neq 0$.

Proof. Consider the $\pi_0(A)$ -algebra $R = \pi_0(A') \otimes_{\pi_0(A)} \pi_0(A'')$. Since it is a nonzero algebra, there exists $\mathfrak{p} \in \mathrm{Spec} \pi_0(A)$ such that $R \otimes_{\pi_0(A)} k(\mathfrak{p}) \neq 0$, where $k(\mathfrak{p})$ is the residue field of $\pi_0 A$ at \mathfrak{p} . Thus, $\pi_0(A') \otimes_{\pi_0(A)} k(\mathfrak{p}) \neq 0$ and $\pi_0(A'') \otimes_{\pi_0(A)} k(\mathfrak{p}) \neq 0$. By Proposition 4.15, the \mathbf{E}_∞ -rings $A' \otimes_A \kappa(\mathfrak{p})$ and $A'' \otimes_A \kappa(\mathfrak{p})$ are nonzero, and thus their relative tensor product over $\kappa(\mathfrak{p})$ is nonzero. Thus, $(A' \otimes_A A'') \otimes_A \kappa(\mathfrak{p}) \neq 0$, so that $A' \otimes_A A'' \neq 0$ as well. \square

Replacing A by $A[t_2^{\pm 1}]$, we also get the following result, which will be important for the next section.

Corollary 4.19. *Let A be a rational noetherian \mathbf{E}_∞ -ring (not necessarily containing a unit in degree two). Let $A', A'' \in \mathrm{CAlg}_{A/}$ be noetherian. Then the following are equivalent:*

- (1) $\pi_{\mathrm{even}}(A') \otimes_{\pi_{\mathrm{even}}(A)} \pi_{\mathrm{even}}(A'') \neq 0$.
- (2) $A' \otimes_A A'' \neq 0$.

Remark 4.20. Given a rational, noetherian \mathbf{E}_∞ -ring A containing a unit in π_2 , we constructed a family of residue fields $\{\kappa(\mathfrak{p})\}_{\mathfrak{p} \in \mathrm{Spec} A}$ that detect nilpotence in the ∞ -category $\mathrm{Mod}(A)$. One could ask if one has in fact a *Bousfield decomposition*. That is, if an A -module M (not necessarily perfect) has the property that $\kappa(\mathfrak{p})_*(M) = 0$ for all $\mathfrak{p} \in \mathrm{Spec} A$, does that force M to be contractible? In the regular and even periodic case, this is known (e.g., [Mat14b, Prop. 2.5]). The analog over the sphere fails: there are noncontractible spectra that smash to zero with every residue field, for instance the Brown-Comenentz dual I of the sphere ([HS99, Appendix B]). We do not know what happens in $\mathrm{Mod}(A)$.

5. A THICK SUBCATEGORY THEOREM

In this section, we use Theorem 4.14 to obtain the classification of thick subcategories of perfect modules over a rational, noetherian \mathbf{E}_∞ -ring.

5.1. Review of the axiomatic argument. Let A be an \mathbf{E}_∞ -ring together with a collection $\{\kappa(\mathfrak{p})_*\}$ of multiplicative homology theories on $\mathrm{Mod}(A)$, satisfying perfect Künneth isomorphisms, such that together they detect nilpotence over A . In this case, it is well-known that they are sufficient to detect thick subcategories as well. We note that these are the same as thick tensor-ideals, since the unit generates $\mathrm{Mod}^\omega(A)$ as a thick subcategory. In particular, consider $M, N \in \mathrm{Mod}^\omega(A)$. We recall:

Theorem 5.1 (Hopkins-Smith-Hovey-Palmieri-Strickland [HPS97, Th. 5.2.2 and Cor. 5.2.3]). *Suppose $M, N \in \mathrm{Mod}^\omega(A)$ are such that whenever $\kappa(\mathfrak{p})_*(M) \neq 0$, then $\kappa(\mathfrak{p})_*(N) \neq 0$ too. Then the thick subcategory that N generates in $\mathrm{Mod}^\omega(A)$ contains M .*

At least under noetherian hypotheses, every thick subcategory $\mathcal{C} \subset \text{Mod}^\omega(A)$ is then determined by a subset of the $\{\kappa(\mathfrak{p})_*\}$ (the $\kappa(\mathfrak{p})_*$ such that there exists $X \in \mathcal{C}$ with $\kappa(\mathfrak{p})_*(X) \neq 0$), and the classification of thick subcategories reduces to the determination of which subsets arise from thick subcategories; or equivalently, which subsets of the $\{\kappa(\mathfrak{p})\}$ arise as $\{\mathfrak{p} : \kappa(\mathfrak{p})_*(M) \neq 0\}$ for some $M \in \text{Mod}^\omega(A)$.

5.2. The even periodic case. The main subtleties of the present section will revolve around the grading. As a result, we start with the simplest case, where the ring contains a unit in degree two. This is a direct consequence of the work of the previous section and the axiomatic argument, Theorem 5.1.

Construction 5.2. Let A be a rational, noetherian \mathbf{E}_∞ -ring containing a unit in degree two. Let $Z \subset \text{Spec}\pi_0 A$ be a specialization-closed subset. We define a thick subcategory $\text{Mod}_Z^\omega(A) \subset \text{Mod}^\omega(A)$ consisting of modules M such that $\pi_0(M) \oplus \pi_1(M)$ is set-theoretically supported on a closed subset of Z .

Construction 5.2 clearly defines thick subcategories of $\text{Mod}^\omega(A)$. We start by noting that they can also be defined in terms of the residue fields of A .

Proposition 5.3. *Let A be a rational, noetherian 2-periodic \mathbf{E}_∞ -ring. Let M be a perfect A -module. Then the following are equivalent, for $\mathfrak{p} \in \text{Spec}\pi_0 A$:*

- (1) $M_{\mathfrak{p}} \neq 0$.
- (2) $\kappa(\mathfrak{p})_*(M) \neq 0$ (where $\kappa(\mathfrak{p}) \in \text{CAlg}_{A/}$ is the residue field of Theorem 4.14).

Proof. Without loss of generality, we can assume that $\pi_0(A)$ is local and that \mathfrak{p} is the maximal ideal of $\pi_0(A)$. Then we need to show that if $\kappa(\mathfrak{p})_*(M) = 0$, then M itself is contractible, a form of Nakayama's lemma. Without loss of generality, we can assume that $\pi_0(A)$ is *complete* local, because the completion is faithfully flat over A [Mat80, Theorem 56, §24].

Let $x_1, \dots, x_n \in \pi_0(A)$ be generators for the maximal ideal \mathfrak{p} . Then it suffices to show that $M/(x_1, \dots, x_n)$, which is the base-change of M to $A/(x_1, \dots, x_n)$, is contractible, because M is (x_1, \dots, x_n) -adically complete. In particular, we may replace A with $A/(x_1, \dots, x_n)$ and thus assume that $\pi_0(A)$ is actually *local artinian*. We thus reduce to this case.

But if $\pi_0(A)$ is local artinian, we know that the map $A \rightarrow \kappa(\mathfrak{p})$ actually *detects nilpotence*: in particular, it cannot annihilate a nonzero perfect A -module (Example 4.5). So, if $\kappa(\mathfrak{p})_*(M) = 0$, then M is contractible. \square

Proposition 5.4. *Let A be a rational, noetherian \mathbf{E}_∞ -ring containing a unit in degree two. Then the thick subcategories of $\text{Mod}^\omega(A)$ are in natural bijection with the specialization-closed subsets of $\text{Spec}\pi_0 A$, via the correspondence given in Construction 5.2.*

Proof. By Theorem 5.1 and Proposition 5.3, it follows that if $M, N \in \text{Mod}^\omega(A)$ and the set-theoretic support of $\pi_0(M) \oplus \pi_1(M)$ contains that of $\pi_0(N) \oplus \pi_1(N)$, then N belongs to the thick subcategory generated by M .

Next, we argue that any closed subset of $\text{Spec}\pi_0 A$ is realized as the support of $\pi_0 M$ for some $M \in \text{Mod}^\omega(A)$. If the closed subset is defined by the ideal $(x_1, \dots, x_n) \in \pi_0(A)$, then we can take $M = A/(x_1, \dots, x_n)$, thanks to Theorem 2.22. This implies that if $Z, Z' \subset \text{Spec}\pi_0 A$ are two distinct specialization-closed subsets, say $Z \setminus Z' \neq \emptyset$ then there exists a module $M \in \text{Mod}^\omega(A)$ which belongs to $\text{Mod}_Z^\omega(A) \setminus \text{Mod}_{Z'}^\omega(A)$. In other words, the map of Construction 5.2 from subsets to thick subcategories is injective.

Finally, if $\mathcal{C} \subset \text{Mod}^\omega(A)$ is a thick subcategory, we let Z be the specialization-closed subset of those $\mathfrak{p} \in \text{Spec}\pi_0(A)$ such that there exists $M \in \mathcal{C}$ with $\kappa(\mathfrak{p})_*(M) \neq 0$. Clearly, $\mathcal{C} \subset \text{Mod}_Z^\omega(A)$. To see equality, fix an arbitrary $M \in \text{Mod}_Z^\omega(A)$. Then there exists a set $\{M_\alpha\}_{\alpha \in S}$ of objects in \mathcal{C} such that the support of $\pi_0 M \oplus \pi_1 M$ is contained in the union of the supports of the $\{\pi_0 M_\alpha \oplus \pi_1 M_\alpha\}$. Therefore, since $\pi_0(A)$ is noetherian, there exists a finite subcollection $S' \subset S$ such that the same conclusion holds, and Theorem 5.1 implies that M belongs to the thick subcategory generated by $\bigoplus_{\alpha \in S'} M_\alpha \in \mathcal{C}$. \square

5.3. Graded rings. In the rest of the section, we will explain how to adapt the argument of Proposition 5.4 to the general case, where we do not assume the existence of a unit in degree two. We begin with a review of some facts about graded rings. We will work with graded rings which are commutative (in the *ungraded* sense) such as π_{even} of an \mathbf{E}_∞ -ring.

Definition 5.5. Let R_* be a commutative, graded ring. The topological space $\text{GrSpec}(R_*)$ consists of the homogeneous prime ideals of R_* . The topology on $\text{GrSpec}(R_*)$ is defined by taking as a basis of open sets the subsets $V(a) \stackrel{\text{def}}{=} \{\mathfrak{p} \in \text{GrSpec}(R_*) : a \notin \mathfrak{p}\}$ for each homogeneous element a . Note that $\text{GrSpec}(R_*) \subset \text{Spec} R_*$ and the inclusion map is continuous (for the usual Zariski topology on the latter).

Given a graded ring R_* , the space $\text{GrSpec}(R_*)$ is also the underlying topological space [LMB00, Ch. 5] of the algebraic stack $(\text{Spec} R_*)/\mathbb{G}_m$, where the \mathbb{G}_m -action on $\text{Spec} R_*$ is given by the grading of R_* . Given a point of $(\text{Spec} R_*)/\mathbb{G}_m$, represented by a map $\text{Spec} k \rightarrow (\text{Spec} R_*)/\mathbb{G}_m$ where k is a field, one obtains a map of graded rings $R_* \rightarrow k[t^{\pm 1}]$ where $|t| = 1$. The kernel of this map is a homogeneous prime ideal of $\text{GrSpec}(R_*)$, which gives the correspondence between points of the stack and $\text{GrSpec}(R_*)$.

Example 5.6. Suppose R_* is nonnegatively graded, i.e., $R_i = 0$ for $i < 0$, and suppose R_0 is a field. Then $\text{GrSpec}(R_*)$ is the union of $\text{Proj}(R_*)$ and one additional point, corresponding to the irrelevant ideal $\bigoplus_{i>0} R_i$.

Definition 5.7. Let R_* be a commutative, graded ring. A collection $\mathfrak{C} \subset \text{GrSpec}(R_*)$ of homogeneous prime ideals of R_* is *closed under specialization* if, whenever $\mathfrak{p} \in \mathfrak{C}$ and $\mathfrak{q} \supset \mathfrak{p}$ is a larger homogeneous prime ideal, then $\mathfrak{q} \in \mathfrak{C}$ too. $\mathfrak{C} \subset \text{GrSpec}(R_*)$ is closed under specialization if and only if it is a union of closed subsets.

We next observe that there is a notion of “support” in the graded setting.

Definition 5.8. Suppose R_* is noetherian and M_* is a finitely generated graded R_* -module. Let $\text{Supp}(M_*)$ denote the collection of all $\mathfrak{p} \in \text{GrSpec}(R_*)$ such that the localization $(M_*)_{\mathfrak{p}}$ does not vanish. Then $\text{Supp}(M_*) \subset \text{GrSpec}(R_*)$ is closed as it is the intersection of the usual support of M_* in $\text{Spec}(R_*)$ with $\text{GrSpec}(R_*) \subset \text{Spec}(R_*)$.

A priori, the construction of the localization $(M_*)_{\mathfrak{p}}$ (which is not a graded R_* -module) is somewhat unnatural. However, it is easy to see that the condition that $(M_*)_{\mathfrak{p}} \neq 0$ is equivalent to the condition that the localization of M_* at the multiplicative subset $S = \{r \in R_* \text{ homogeneous} : r \notin \mathfrak{p}\}$ should not vanish, and this latter localization is naturally a graded R_* -module.

We will next need the notion of a graded-local ring.

Definition 5.9. R_* is *graded-local* if it has a unique maximal homogeneous ideal. R_* is a *graded field* if either:

- (1) $R_* = k$, concentrated in degree zero, where k is a field.
- (2) $R_* = k[u, u^{-1}]$ where $|u| > 0$ and k is a field.

It is easy to see that a graded ring is a graded field if and only if the zero ideal is a maximal homogeneous ideal. In fact, this condition implies that any homogeneous element is either zero or a unit. As a result, given any $\mathfrak{p} \in \text{GrSpec}(R_*)$, we can form the graded R_* -algebra $R_{*[\mathfrak{p}]} / \mathfrak{p}$, where $R_{*[\mathfrak{p}]}$ is the localization of R_* at the set of homogeneous elements not in \mathfrak{p} . This is a graded field.

We will next need to understand a little about the interaction between homogeneous and inhomogeneous prime ideals.

Construction 5.10. Let $\mathfrak{q} \in \text{Spec}(R_*)$ be a prime ideal (not assumed homogeneous). We define a homogeneous prime ideal $\mathfrak{p} \in \text{GrSpec}(R_*)$ such that $x \in \mathfrak{p}$ if and only if each homogeneous component of x belongs to \mathfrak{q} . Clearly, $\mathfrak{p} \subset \mathfrak{q}$ is the maximal homogeneous ideal contained in \mathfrak{q} .

In the language of stacks, we have a quotient map $\text{Spec} R_* \rightarrow (\text{Spec} R_*)/\mathbb{G}_m$, which induces a map on points $\text{Spec} R_* \rightarrow \text{GrSpec}(R_*)$. This map sends $\mathfrak{q} \mapsto \mathfrak{p}$ as above.

Finally, to use both graded and ungraded techniques, we need the following construction.

Construction 5.11. Let R_* be a commutative, graded noetherian ring. Let R'_* be the graded ring $R_*[u^{\pm 1}]$ where $|u| = 1$. Then graded R'_* -modules are canonically in correspondence with *ungraded* R_* -modules. For example, let $\mathfrak{q} \in \text{Spec}(R_*)$ be a prime ideal, not assumed homogeneous. We let $k(\mathfrak{q})_*$ denote the graded R'_* -module corresponding to ungraded R_* -module which is the residue field of R_* at \mathfrak{q} .

Given an ungraded R_* -module M , we can form a graded R'_* -module M'_* as in Construction 5.11 and restrict to get a graded R_* -module (still denoted M'_*), such that $M'_n = M$ for every n . The following elementary lemma will be crucial in the next subsection.

Lemma 5.12. *Let R_* be a graded, commutative noetherian ring. Let $R'_* = R_*[u^{\pm 1}]$ where $|u| = 1$. Let $\mathfrak{q} \in \text{Spec} R_*$ and let $\mathfrak{p} \in \text{GrSpec}(R'_*)$ be the homogeneous part. Define the graded R'_* -algebras $k(\mathfrak{p})_*, k(\mathfrak{q})_*$ as in Construction 5.11. Then $k(\mathfrak{p})_* \otimes_{R'_*} k(\mathfrak{q})_* \neq 0$.*

Proof. We can assume without loss of generality that R_* is graded-local with maximal homogeneous ideal \mathfrak{p} . After replacing R_* with R_*/\mathfrak{p} , we can assume $\mathfrak{p} = 0$. In this case, R_* is a graded field so that the tensor product of any two nonzero graded R_* -modules (e.g., $k(\mathfrak{p})_*, k(\mathfrak{q})_*$, under pull-back from $R_* \rightarrow R'_*$) is nonzero. \square

5.4. The thick subcategory theorem. Let A be a rational, noetherian \mathbf{E}_∞ -ring. The purpose of this subsection is to give the proof that thick subcategories of $\text{Mod}^\omega(A)$ correspond to specialization-closed subsets of $\text{GrSpec}(\pi_{\text{even}}(A))$, without assuming the existence of a unit in degree two. We first state formally the map that realizes the correspondence.

Definition 5.13. Let $M \in \text{Mod}^\omega(A)$. We define the *support* $\text{Supp}(M)$ to be the support (in the sense of Definition 5.8) of the graded $\pi_{\text{even}}(A)$ -module $\pi_*(M)$. We will denote this by $\text{Supp}M$; it is a subset of $\text{GrSpec}(\pi_{\text{even}}(A))$. Given a specialization-closed subset $Z \subset \text{GrSpec}(\pi_{\text{even}}(A))$, we define $\text{Mod}_Z^\omega(A) \subset \text{Mod}^\omega(A)$ to be the full subcategory spanned by those perfect A -modules M with $\text{Supp}M \subset Z$.

Theorem 5.14. *Let A be a rational, noetherian \mathbf{E}_∞ -ring. Then the construction $Z \mapsto \text{Mod}_Z^\omega(A)$ defines a correspondence between the thick subcategories of $\text{Mod}^\omega(A)$ and specialization-closed subsets of $\text{GrSpec}(\pi_{\text{even}}(A))$.*

The primary goal of this section is to give a proof of Theorem 5.14, which we already did (in Proposition 5.4) in case $\pi_2(A)$ contains a unit.

To begin with, we will need to discuss residue fields for A . Let A be a noetherian rational \mathbf{E}_∞ -ring and $\mathfrak{p} \subset \pi_{\text{even}}(A)$ a homogeneous prime ideal. We form the \mathbf{E}_∞ -ring $A' = A[t_2^{\pm 1}]$ and we then have

$$\pi_0 A' \simeq \pi_{\text{even}} A.$$

In particular, \mathfrak{p} becomes a prime ideal of $\pi_0 A'$. As a result, in view of Theorem 4.14, we can construct a residue field $\kappa(\mathfrak{p})$ of A' at \mathfrak{p} and we obtain maps $A \rightarrow A' \rightarrow \kappa(\mathfrak{p})$. Rather than considering the $\{\kappa(\mathfrak{p})\}$ as \mathbf{E}_∞ - A' -algebras, we consider them as \mathbf{E}_∞ - A -algebras. They satisfy a perfect Künneth isomorphism as homology theories on $\text{Mod}(A)$, as before.

Lemma 5.15. *Let $\mathfrak{q} \subset \pi_{\text{even}}(A)$ be an inhomogeneous prime ideal and let $\mathfrak{p} \subset \pi_{\text{even}}(A)$ be the homogeneous part. Let $\kappa(\mathfrak{q}), \kappa(\mathfrak{p}) \in \text{CAlg}_{A'}$ denote the respective residue fields, which we regard as \mathbf{E}_∞ - A -algebras under $A \rightarrow A'$. Then $\kappa(\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}) \neq 0$.*

Proof. This follows from Corollary 4.19 and Lemma 5.12. \square

As a result, we can now show that the residue fields $\kappa(\mathfrak{p}) \in \text{CAlg}_{A'}$ for $\mathfrak{p} \in \text{GrSpec}(\pi_{\text{even}}(A))$ suffice to detect nilpotence.

Theorem 5.16. *The $\{\kappa(\mathfrak{p})\}$, as \mathfrak{p} ranges over the homogeneous prime ideals in $\text{Spec} \pi_{\text{even}}(A)$, detect nilpotence over A .*

This is not an immediate consequence of Theorem 4.14 applied to A' , because the homogeneous \mathfrak{p} do not exhaust all the prime ideals of $\pi_0(A')$. In other words, the $\kappa(\mathfrak{p})$ in question do not detect nilpotence over A' .

Proof. We know that the $\{\kappa(\mathfrak{q})\} \subset \mathrm{CAlg}_{A'}$ for $\mathfrak{q} \in \mathrm{Spec} \pi_{\mathrm{even}} A = \pi_0 A'$ detect nilpotence over A' (Theorem 4.14), and thus over A . Thus, in order to prove that the $\{\kappa(\mathfrak{p})\}$ for \mathfrak{p} ranging over the *homogeneous* prime ideals detect nilpotence over A , we appeal to the third part of Proposition 4.6 and Lemma 5.15. \square

We note now that if $M \in \mathrm{Mod}^\omega(A)$, then the support of M in $\mathrm{GrSpec}(\pi_{\mathrm{even}}(A))$ is equivalently the set of $\mathfrak{p} \in \mathrm{GrSpec}(\pi_{\mathrm{even}}(A))$ such that $M \otimes_A \kappa(\mathfrak{p}) \neq 0$; this is a consequence of Proposition 5.3.

Proof of Theorem 5.14. In particular, it now follows formally (via the axiomatic argument given in Theorem 5.1, together with the noetherianness of $\mathrm{GrSpec}(\pi_{\mathrm{even}}(A))$ from Theorem 5.16 that a thick subcategory of $\mathrm{Mod}^\omega(A)$ is determined by a subcollection of the $\{\kappa(\mathfrak{p})\}$ as \mathfrak{p} ranges over the homogeneous prime ideals of $\pi_{\mathrm{even}}(A)$. It remains to determine what subsets are allowed to arise.

We will show that those subsets are precisely those which are closed under specialization. To see this, we need to show that every closed subset of $\mathrm{GrSpec}(\pi_{\mathrm{even}}(A))$ (associated to a homogeneous ideal $I \subset \pi_{\mathrm{even}}(A)$) can be realized as the support of some M , but this follows by forming $A/(x_1, \dots, x_n)$ where $x_1, \dots, x_n \in \pi_{\mathrm{even}}(A)$ generate I . In particular, this completes the proof of Theorem 5.14. \square

6. GALOIS GROUPS

Let A be an \mathbf{E}_∞ -ring such that $\pi_0(A)$ has no nontrivial idempotents. In [Mat14a], we introduced the *Galois group* $\pi_1 \mathrm{Mod}(A)$ of A , a profinite group defined “up to conjugacy” (canonically as a profinite *groupoid*), by developing a version of the étale fundamental group formalism. The Galois group $\pi_1 \mathrm{Mod}(A)$ has the property that if G is a finite group, then to give a continuous group homomorphism $\pi_1 \mathrm{Mod}(A) \rightarrow G$ is equivalent to giving a faithful G -Galois extension of A in the sense of Rognes [Rog08]. More generally, we introduced ([Mat14a, Def. 6.1]) the notion of a *finite cover* of an \mathbf{E}_∞ -ring A , as a homotopy-theoretic version of the classical notion of a finite étale algebra over a commutative ring. A continuous action of $\pi_1 \mathrm{Mod}(A)$ on a finite set is equivalent to a finite cover of the \mathbf{E}_∞ -ring A . The Galois group can be a fairly sensitive invariant of \mathbf{E}_∞ -rings; for instance ([Mat14a, Ex. 7.21]) two different \mathbf{E}_∞ -structures on the same \mathbf{E}_1 -ring can yield different Galois groups, and computing it appears to be a subtle problem in general. Here, we will show that the Galois group is much less sensitive over the rational numbers, under noetherian hypotheses.

The Galois group comes with a surjection

$$(11) \quad \pi_1 \mathrm{Mod}(A) \twoheadrightarrow \pi_1^{\mathrm{et}} \mathrm{Spec} \pi_0(A),$$

since every algebraic Galois cover of $\mathrm{Spec} \pi_0(A)$ can be realized topologically. More generally, to every finite étale $\pi_0(A)$ -algebra A'_0 one can canonically associate $A' \in \mathrm{CAlg}_{A'}$ such that $\pi_0 A' \simeq A'_0$ and such that $\pi_k A' \simeq A'_0 \otimes_{\pi_0 A} \pi_k A$ [Lur14, §7.5]. This yields a full subcategory of the category of finite covers which corresponds to the above surjection. In general, however, it is an insight of Rognes that the above surjection has a nontrivial kernel: that is, there exist finite covers that do not arise algebraically in this fashion. A basic example is the complexification map $KO \rightarrow KU$.

In [Mat14a], we computed Galois groups in certain instances. Our basic ingredient ([Mat14a, Th. 6.30]) was a strengthening of work of Baker-Richter [BR08] to show that the Galois theory is entirely algebraic for even periodic \mathbf{E}_∞ -rings with *regular* π_0 , using the theory of residue fields. Over the rational numbers, the methods of the present paper enable one to construct these “residue fields” without regularity assumptions. In particular, we will show in this section that, for noetherian rational \mathbf{E}_∞ -rings, the computation of the Galois group can be reduced to a problem of pure algebra. (For instance, we will show that (11) is an isomorphism if A contains a unit in degree two.) In general, (11) will not be an isomorphism, because over \mathbb{Q} , it is permissible to adjoin square roots of invertible elements in homotopy in degrees divisible by four, for instance. But we will see that such issues of grading are the *only* failure of (11) to be an isomorphism.

6.1. Review of invariance properties. To obtain the results of the present section, we will need some basic tools for working with Galois groups, which will take the form of the “invariance results” of [Mat14a]. For example, we will need to know that killing a nilpotent degree zero class

does not affect the Galois group. For convenience, we will assume that all \mathbf{E}_∞ -rings A considered in this section have no nontrivial idempotents in π_0 , so that we can speak about a Galois group.

Theorem 6.1. *Let A be a rational \mathbf{E}_∞ -ring and let $x \in \pi_0 A$ be a nilpotent element. Then the map $A \rightarrow A//x$ induces an isomorphism on Galois groupoids.*

Proof. This is [Mat14a, Theorem 8.13], for the map $\mathbb{Q}[[t]] \rightarrow A$ sending $t \mapsto x$. \square

Proposition 6.2. *Let A be a rational \mathbf{E}_∞ -ring and let $x \in \pi_{-1} A$ be a class. Then the map*

$$A \rightarrow A//x \simeq A \otimes_{\mathrm{Sym}^* \mathbb{Q}[-1]} \mathbb{Q},$$

obtained by coning off x , induces a surjection on Galois groups.

Proof. By [Mat14a, §8.1], it suffices to show that the map $C^*(S^1; \mathbb{Q}) \simeq \mathrm{Sym}^* \mathbb{Q}[-1] \rightarrow \mathbb{Q}$ is *universally connected*: that is, for any $A \in \mathrm{CAlg}_{C^*(S^1; \mathbb{Q})}$, the natural map $A \rightarrow A \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q}$ induces an isomorphism on Idem .

Since $C^*(S^1; \mathbb{Q}) \rightarrow \mathbb{Q}$ admits descent, the set $\mathrm{Idem}(A)$ of idempotents in A is the equalizer of the two maps

$$A \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q} \rightrightarrows A \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q} \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q} \simeq (A \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q}) [t].$$

This is a reflexive equalizer, and one of the maps is the natural inclusion

$$A \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q} \rightarrow (A \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q}) [t],$$

which induces an isomorphism on idempotents. It follows that all the maps in the reflexive equalizer are isomorphisms and thus the two forward maps are equal, proving that $A \rightarrow A \otimes_{C^*(S^1; \mathbb{Q})} \mathbb{Q}$ induces an isomorphism on idempotents. \square

6.2. The periodic case. We are now ready to show (Theorem 6.4 below) that the Galois theory of a noetherian rational \mathbf{E}_∞ -ring containing a degree two unit is algebraic.

Lemma 6.3. *Let A be a rational, noetherian \mathbf{E}_∞ -ring containing a unit in π_2 such that $\pi_0 A$ is local artinian. Then the Galois theory of A is algebraic.*

Proof. The strategy is to imitate the proof of Theorem 4.14, while cognizant of the invariance results for Galois groups reviewed in the previous subsection. Namely, we not only showed that A had a residue field, but we constructed it via a specific recipe. Let k be the residue field of $\pi_0 A$.

In proving Theorem 4.14 (that is, in the course of the proof of Proposition 4.12), we first formed a sequence of rational, noetherian, \mathbf{E}_∞ -rings with artinian π_0 ,

$$A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_\infty = \varinjlim A_i,$$

such that:

- (1) A_{i+1} is obtained from A_i by attaching 1-cells along a finite number of nilpotent elements in $\pi_0(A_i)$.
- (2) All the $\pi_0(A_i)$ are local artinian rings with residue field k , and each map $\pi_0(A_i) \rightarrow \pi_0(A_{i+1})$ annihilates the maximal ideal.

By Theorem 6.1, at no finite stage do we change the Galois group; each map $A \rightarrow A_i$ induces an isomorphism on Galois groups. Now, by [Mat14a, Th. 6.21], the Galois group is compatible with filtered colimits and therefore $A \rightarrow A_\infty$ induces an isomorphism on Galois groups.¹

Now, the \mathbf{E}_∞ -ring A_∞ has the properties of Proposition 4.9: it has a unit in degree two, its π_0 is isomorphic to k , and π_1 is countably dimensional. We showed in the proof of Proposition 4.9 that by killing degree -1 cells repeatedly and forming countable colimits, and repeating countably many times, we could start with A_∞ and reach $k[t_2^{\pm 1}]$. It follows by Proposition 6.2 (along with the compatibility of Galois groups and filtered colimits, again) that the map

$$A_\infty \rightarrow k[t_2^{\pm 1}],$$

¹In fact, all we need for the proof of this lemma is that $A \rightarrow A_\infty$ induces a *surjection* on Galois groups. This does not require the obstruction theory used in proving [Mat14a, Th. 6.21], and is purely formal.

induces a *surjection* on Galois groups. But the Galois group of $k[t_2^{\pm 1}]$ is algebraic (i.e., $\text{Gal}(\bar{k}/k)$) in view of the Künneth isomorphism [Mat14a, Prop. 6.28], so the Galois group of A_∞ must be bounded by $\text{Gal}(\bar{k}/k)$, and therefore that of A must be, too. \square

We can now prove the main result of the present subsection.

Theorem 6.4. *Let A be a rational, noetherian \mathbf{E}_∞ -ring containing a unit in degree two. Then the Galois theory of A is algebraic, i.e., $\pi_1 \text{Mod}(A) \simeq \pi_1^{\text{ét}} \text{Spec} \pi_0(A)$.*

Proof. Fix a finite cover $A \rightarrow A'$ of \mathbf{E}_∞ -rings. We need to show that A' is *flat* over A : that is, the natural map $\pi_0(A) \rightarrow \pi_0(A')$ is flat, and the map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(A') \rightarrow \pi_*(A')$ is an isomorphism. This is a local question, so we may assume that $\pi_0(A)$ is a local noetherian ring. Moreover, by completing A at the maximal ideal $\mathfrak{m} \subset \pi_0 A$, we may assume that $\pi_0 A$ is *complete*; we may do this because the completion of any noetherian local ring is faithfully flat over it.

Let k be the residue field of the (discrete) commutative ring $\pi_0 A$. The étale fundamental group of $\text{Spec} \pi_0 A$ is naturally isomorphic to that of $\text{Spec} k$, via the inclusion $\text{Spec} k \hookrightarrow \text{Spec} \pi_0 A$ as the closed point, since $\pi_0 A$ is a complete local ring. Let $x_1, \dots, x_n \in \pi_0 A$ be generators for the maximal ideal. Consider the tower of \mathbf{E}_∞ - A -algebras

$$\cdots \rightarrow A/(x_1^3, \dots, x_n^3) \rightarrow A/(x_1^2, \dots, x_n^2) \rightarrow A/(x_1, \dots, x_n),$$

whose inverse limit is given by A itself (by completeness). Observe that the maps at each stage are not uniquely determined. For instance, to give a map $A/(x_1^2, \dots, x_n^2) \rightarrow A/(x_1, \dots, x_n)$ amounts to giving nullhomotopies of each of x_1^2, \dots, x_n^2 in $A/(x_1, \dots, x_n)$, and there are many possible choices of nullhomotopies. One has to make choices at each stage.

Denote the \mathbf{E}_∞ -algebras in this tower by $\{A_m\}$. As a result, the equivalence $A \simeq \varprojlim A_m$ leads to a fully faithful imbedding

$$\text{Mod}^\omega(A) \subset \varprojlim \text{Mod}^\omega(A_m),$$

from the ∞ -category $\text{Mod}^\omega(A)$ of perfect A -modules into the homotopy limit of the ∞ -categories $\text{Mod}^\omega(A_m)$ of perfect A_m -modules. As discussed in [Mat14a, §7.1], this implies that if we show that the Galois group of each A_m is equivalent to that of k (i.e., is algebraic), then the Galois group of A itself is forced to be algebraic. This, however, is precisely what we proved in Lemma 6.3 above. \square

6.3. The general case. In the previous parts of this section, we showed that the Galois theory of a rational noetherian \mathbf{E}_∞ -ring A containing a unit in π_2 was entirely algebraic. In this subsection, we will explain the modifications needed to handle the case where we do not have a unit in π_2 ; in this case, the structure of the entire homotopy ring $\pi_* A$ (rather than simply $\pi_0 A$) intervenes. We will begin with some generalities from [BR07] which, incidentally, shed further light on Galois groups of general \mathbf{E}_∞ -rings.

Let R_* be a commutative, \mathbb{Z} -graded ring (*not* graded-commutative!), such as $\pi_{\text{even}}(A)$ for $A \in \text{CAlg}$. We start by setting up a Galois formalism for R_* that takes into account the grading.

Definition 6.5. A *graded finite étale* R_* -algebra is a commutative, graded R_* -algebra R'_* such that, as underlying commutative rings, the map $R_* \rightarrow R'_*$ is finite étale.

We list two fundamental examples:

Example 6.6. Given a finite étale R_0 -algebra R'_0 , then one can build from this a graded finite étale R_* -algebra via $R'_* \stackrel{\text{def}}{=} R'_0 \otimes_{R_0} R_*$.

Example 6.7. Let $R_* = \mathbb{Z}[1/n, x_n^{\pm 1}]$ where $|x_n| = n$. Then the map $R_* \rightarrow R'_* = \mathbb{Z}[1/n, y_1^{\pm 1}]$, $x_n \mapsto y_1^n$ is graded finite étale. In other words, one can adjoin n th roots of invertible generators in degrees divisible by n , over a $\mathbb{Z}[1/n]$ -algebra.

Consider the category \mathcal{C}_{R_*} of graded finite étale R_* -algebras and graded R_* -algebra homomorphisms. We start by observing that it is opposite to a Galois category. One can formulate this in the following manner. The grading on R_* determines an action of the multiplicative group \mathbb{G}_m on

$\mathrm{Spec}R_*$, in such a manner that to give a quasi-coherent sheaf on the quotient stack $(\mathrm{Spec}R_*)/\mathbb{G}_m$ is equivalent to giving a graded R_* -module. To give a finite étale cover of the quotient stack $(\mathrm{Spec}R_*)/\mathbb{G}_m$ is equivalent to giving a graded R_* -algebra which is finite étale over R_* . In other words, graded finite étale R_* -algebras are equivalent to finite étale covers of the *stack* $\mathrm{Spec}R_*/\mathbb{G}_m$.

Definition 6.8. We define the *graded étale fundamental group* $\pi_1^{\mathrm{et}, \mathrm{gr}} \mathrm{Spec}R_*$ to be the étale fundamental group of the stack $(\mathrm{Spec}R_*)/\mathbb{G}_m$.

Example 6.9. Suppose R_* contains a unit in degree 1. In this case, $R_* \simeq R_0 \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$ where $|t| = 1$. In particular, the quotient stack $(\mathrm{Spec}R_*)/\mathbb{G}_m$ is simply $\mathrm{Spec}R_0$, so the graded étale fundamental group of R_* is the fundamental group of $\mathrm{Spec}R_0$.

Now, let $A_{\mathrm{even}} = \pi_{\mathrm{even}}(A)$, constructed with the degree halved (so that $(A_{\mathrm{even}})_1 = \pi_2(A)$) and fix a graded, finite étale A_{even} -algebra A'_* . We will construct an \mathbf{E}_{∞} - A -algebra A' equipped with an isomorphism $\pi_{\mathrm{even}}A' \simeq A'_*$ which is a finite cover of A , generalizing the construction that starts with a finite étale π_0A -algebra and obtains a finite cover of A . As a result, we will obtain:

Theorem 6.10 (Baker-Richter [BR07]). *There is a natural fully faithful imbedding from the category of graded, finite étale A_{even} -algebras into the category of finite covers of A in \mathbf{E}_{∞} -ring spectra.*

Dually, we obtain surjections of profinite groups

$$(12) \quad \pi_1 \mathrm{Mod}(A) \twoheadrightarrow \pi_1^{\mathrm{et}, \mathrm{gr}}(\mathrm{Spec}A_{\mathrm{even}}) \twoheadrightarrow \pi_1^{\mathrm{et}}(\mathrm{Spec}\pi_0A),$$

refining the surjection (11). Since our formulation of the (essentially same) result is slightly different from that of Baker-Richter, we give the deduction from their statement below.

Proof. Start with a G -Galois object A'_* in the category of graded, finite étale A_{even} -algebras (for G a finite group). Then A'_* is a projective A_{even} -module. Let $\tilde{A}'_* = A'_* \otimes_{A_{\mathrm{even}}} \pi_*(A)$ be the graded-commutative algebra obtained by tensoring up; it acquires a G -action. Moreover, it is a finitely generated, projective $\pi_*(A)$ -module, and the map

$$\tilde{A}'_* \otimes_{\pi_*(A)} \tilde{A}'_* \rightarrow \prod_G \tilde{A}'_*,$$

given by all the twisted multiplications $a_1 \otimes a_2 \mapsto a_1 g(a_2)$ for $g \in G$, is an isomorphism. By [BR07, Theorem 2.1.1], we can construct an object $A' \in \mathrm{CAlg}_{A/}$ with a G -action such that A' is a G -Galois extension of A and realizes the above map $\pi_*(A) \rightarrow \tilde{A}'_*$ on homotopy groups.

We can now describe the universal property of A' as an \mathbf{E}_{∞} - A -algebra. In fact, we claim that for any \mathbf{E}_{∞} - A -algebra B , we have a natural homotopy equivalence

$$(13) \quad \mathrm{Hom}_{\mathrm{CAlg}_{A/}}(A', B) = \mathrm{Hom}_{A_{\mathrm{even}}}(A'_*, \pi_{\mathrm{even}}(B)),$$

so in particular the left-hand-side is discrete. But by Galois descent, this is also the G -fixed points of the set of maps

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CAlg}_{A'}}(A' \otimes_A A', B \otimes_A A') &\simeq \prod_G \mathrm{Idem}(B \otimes_A A') \\ &= \mathrm{Hom}_{A'_*}(A'_* \otimes_{\pi_{\mathrm{even}}(A)} A'_*, \pi_{\mathrm{even}}(B) \otimes_{\pi_{\mathrm{even}}(A)} A'_*), \end{aligned}$$

since A' has homotopy groups which are flat over $\pi_*(A)$ and since $A'_* \otimes_{\pi_{\mathrm{even}}(A)} A'_* \simeq \prod_G A'_*$. But using the algebraic form of Galois descent, we get that

$$\mathrm{Hom}_{A'_*}(A'_* \otimes_{\pi_{\mathrm{even}}(A)} A'_*, \pi_{\mathrm{even}}(B) \otimes_{\pi_{\mathrm{even}}(A)} A'_*)^G = \mathrm{Hom}_{A_{\mathrm{even}}}(A'_*, \pi_{\mathrm{even}}(B)),$$

so we get (13) as claimed.

This imbeds the Galois objects in the category of graded, finite étale $\pi_{\mathrm{even}}(A)$ -algebras *fully faithfully* (by (13)) in the category of finite covers of the \mathbf{E}_{∞} -ring. To associate a finite cover to any graded, finite étale $\pi_{\mathrm{even}}(A)$ -algebra, one now uses Galois descent: the Galois objects can be used to split any finite étale algebra object. Full faithfulness on these more general covers can now be checked locally, using descent. \square

With this in mind, we can state and prove our main result.

Theorem 6.11. *Let A be a noetherian, rational \mathbf{E}_∞ -ring. Then the natural map $\pi_1 \text{Mod}(A) \rightarrow \pi_1^{\text{et,gr}}(\text{Spec} \pi_{\text{even}}(A))$ is an isomorphism of profinite group(oid)s.*

Proof. We have already proved this result if A has a unit in degree two, thanks to Theorem 6.4 (see also Example 6.9). We want to claim that for any A satisfying the conditions of this result, the functor from graded, finite étale $\pi_{\text{even}}(A)$ -algebras to finite covers of the \mathbf{E}_∞ -ring A is an equivalence of categories $\mathcal{C}_1 \simeq \mathcal{C}_2$. We already know that the functor $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ is fully faithful.

Both categories depend functorially on A and have a good theory of descent via the base-change of \mathbf{E}_∞ -rings $\mathbb{Q} \rightarrow \mathbb{Q}[t_2^{\pm 1}]$, where $\mathbb{Q}[t_2^{\pm 1}]$ is the free rational \mathbf{E}_∞ -ring on an invertible degree two class. By descent theory, we can thus reduce to the case of a $\mathbb{Q}[t_2^{\pm 1}]$ -algebra, for which we have proved the result in Theorem 6.4. \square

7. THE PICARD GROUP

7.1. Generalities. Another natural invariant that one might attempt to study using the theory of residue fields is the Picard group. Recall that the *Picard group* of an \mathbf{E}_∞ -ring A , denoted $\text{Pic}(A)$, is the group of isomorphism classes of \otimes -invertible A -modules. In fact, these techniques were originally introduced in [HMS94] in the study of the $K(n)$ -local Picard group. It is known that if A is an \mathbf{E}_∞ -ring which is even periodic and whose π_0 is regular, then the Picard group is algebraic ([BR05]).

In this paper, we have given global descriptions (in terms of algebra) of both the Galois theory and the thick subcategories of $\text{Mod}(A)$, for A a noetherian rational \mathbf{E}_∞ -ring. We do not know if it is possible to describe the Picard group of rational \mathbf{E}_∞ -rings in a similar global manner. However, the following example shows that any answer will be necessarily more complicated.

Example 7.1. Consider the \mathbf{E}_∞ -ring $A = \text{Sym}^* \mathbb{Q}[-1] \otimes \mathbb{Q}[\epsilon]/\epsilon^2$, which is obtained from the ring of “dual numbers” by freely adding a generator in degree -1 . By the results of Section 3.3, we have

$$\text{Mod}(A) \subset \text{Loc}_{S^1}(\text{Mod}(\mathbb{Q}[\epsilon]/\epsilon^2)),$$

that is, to give an A -module is equivalent to giving a $\mathbb{Q}[\epsilon]/\epsilon^2$ -module together with an automorphism whose action on homotopy groups is ind-unipotent.

For example, we might consider the $\mathbb{Q}[\epsilon]/\epsilon^2$ -module $\mathbb{Q}[\epsilon]/\epsilon^2$ and equip it with the automorphism given by $1 + r\epsilon$, for any $r \in \mathbb{Q}$. For any $r \in \mathbb{Q}$, this defines an A -module M_r . Since the underlying $\mathbb{Q}[\epsilon]/\epsilon^2$ -module is invertible, it follows that $M_r \in \text{Pic}(\text{Mod}(A))$. Moreover, $M_r \otimes M_{r'} \simeq M_{r+r'}$ (by composing automorphisms). This shows that there is a copy of \mathbb{Q} inside the Picard group of $\text{Mod}(A)$ that one does not see from the homotopy groups of A . In fact, one sees easily that $\text{Pic}(A) \simeq \mathbb{Z} \oplus \mathbb{Q}$.

Nonetheless, we will be able to obtain a weak partial result about the Picard groups of noetherian rational \mathbf{E}_∞ -rings in comparison to the algebraic analog. We will first need some algebraic preliminaries.

Definition 7.2. Given a graded-commutative ring R_* , the category of graded R_* -modules has a symmetric monoidal structure (the graded tensor product). We let $\text{Pic}(R_*)$ denote the Picard group of the category of graded R_* -modules, i.e., the group of isomorphism classes of invertible graded R_* -modules.

Note that any invertible graded R_* -module must be flat, since tensoring with it is an autoequivalence (hence exact).

Proposition 7.3. *Suppose R_* is a graded-commutative ring such that R_2 contains a unit and R_0 is a noetherian ring, and R_1 is a finitely generated R_0 -module. Suppose R_0 has no nontrivial idempotents. Then $\text{Pic}(R_*) = \mathbb{Z}/2 \oplus \text{Pic}(R_0)$, where the $\mathbb{Z}/2$ comes from the shift of R_* .*

Proof. It suffices to show that if R_0 is local, then $\text{Pic}(R_*) = \mathbb{Z}/2$. We prove this first if R_0 is a field k , so there is a map $R_* \rightarrow k[t_2^{\pm 1}]$ of graded rings with nilpotent kernel. Let M_* be an invertible R_* -module. Then $M \otimes_{R_*} k[t_2^{\pm 1}]$ is either $k[t_2^{\pm 1}]$ or its shift since $\text{Pic}(k[t_2^{\pm 1}]) \simeq \mathbb{Z}/2$; assume the former without loss of generality. Choose a homogeneous $\bar{x} \in (M_* \otimes_{R_*} k[t_2^{\pm 1}])_0$ which is a generator

and lift it to a homogeneous element $x \in M_0$. We then obtain a map $R_* \rightarrow M_*$ which induces an isomorphism after tensoring with $k[t_2^{\pm 1}]$. By Nakayama's lemma, one sees that it is surjective. Let K_* be the kernel. Then the short exact sequence

$$0 \rightarrow K_* \rightarrow R_* \rightarrow M_* \rightarrow 0$$

has the property that

$$0 \rightarrow K_* \otimes_{R_*} k[t_2^{\pm 1}] \rightarrow k[t_2^{\pm 1}] \rightarrow k[t_2^{\pm 1}] \rightarrow 0$$

is still exact, by flatness of M_* , and it shows that $K_* = 0$ by Nakayama's lemma.

If R_0 is not a field k , then we can consider the maximal ideal $\mathfrak{m} \subset R_0$ and consider the map $R_* \rightarrow R_*/R_*\mathfrak{m}$ to reduce to this case. Using a similar argument with Nakayama's lemma, and the fact that the Picard group of $R_*/R_*\mathfrak{m}$ is $\mathbb{Z}/2$, we find that the Picard group of R_* is $\mathbb{Z}/2$ as well. \square

Recall that if A is an \mathbf{E}_∞ -ring, one has the following basic construction.

Construction 7.4 ([BR05]). There is an inclusion $\mathrm{Pic}(\pi_*(A)) \rightarrow \mathrm{Pic}(A)$ which sends an invertible graded $\pi_*(A)$ -module M_* to an invertible A -module M (which is uniquely determined) with $\pi_*(M) \simeq M_*$. The image consists of those invertible A -modules M such that $\pi_*(M)$ is a *flat* $\pi_*(A)$ -module.

Elements in the image of the map $\mathrm{Pic}(\pi_*(A)) \rightarrow \mathrm{Pic}(A)$ are said to be *algebraic*; if every element is algebraic, then the Picard group itself is said to be algebraic. There are many cases in which the Picard group of an \mathbf{E}_∞ -ring can be shown to be algebraic. For instance, if A is even periodic with regular noetherian π_0 , or if A is connective, then it is known [BR05] that the Picard group of A is algebraic (see also [MS14, §2.4.6]). Example 7.1 shows that the Picard group of a rational noetherian \mathbf{E}_∞ -ring need not be algebraic. Our main result (Theorem 7.7), however, implies that any *torsion* in the Picard group is necessarily algebraic.

To prove this result, we will need to use some techniques from [MS14] which we review briefly here. Recall ([MS14, Def. 2.2.1]) that the Picard group $\mathrm{Pic}(A)$ is the group of connected components of a connective spectrum $\mathbf{pic}(A)$ called the *Picard spectrum* of A . The infinite loop space $\Omega^\infty \mathbf{pic}(A)$ is associated to the symmetric monoidal ∞ -groupoid of invertible A -modules (under tensor product). The use of $\mathbf{pic}(A)$ (as opposed to simply $\mathrm{Pic}(A)$) is critical when one wishes to appeal to descent-theoretic techniques.

We now outline a basic descent-theoretic technique in the study of Picard groups of ring spectra.

Construction 7.5 (Compare [MS14, §3]). Let $A \rightarrow B$ be a morphism of \mathbf{E}_∞ -rings which admits descent. In this case, we can obtain an expression for the ∞ -category $\mathrm{Mod}(A)$ as the totalization

$$\mathrm{Mod}(A) \simeq \mathrm{Tot} \left(\mathrm{Mod}(B) \rightrightarrows \mathrm{Mod}(B \otimes_A B) \rightrightarrows \dots \right),$$

by descent theory [Mat14a, §3]. As a result, one obtains an expression for the spectrum $\mathbf{pic}(A)$,

$$(14) \quad \mathbf{pic}(A) = \tau_{\geq 0} \mathrm{Tot} \left(\mathbf{pic}(B^{(\otimes \bullet + 1)}) \right),$$

and a resulting homotopy spectral sequence

$$(15) \quad E_2^{s,t} = H^s \left(\pi_t \mathbf{pic}(B^{(\otimes \bullet + 1)}) \right) \implies \pi_{t-s} \mathbf{pic}(A), \quad t - s \geq 0.$$

We recall, moreover, that for any \mathbf{E}_∞ -ring A , we have natural isomorphisms $\pi_1(\mathbf{pic}(A)) \simeq (\pi_0 A)^\times$ and $\pi_t(\mathbf{pic}(A)) \simeq \pi_{t-1}(A)$ for $t \geq 2$. In particular, the descent spectral sequence (15), for $t \geq 2$, has the same E_2 -page as the usual Adams spectral sequence. If $A \rightarrow B$ admits descent, it follows from [MS14, Comparison Tool 5.2.4] together with the analogous result for the $A \rightarrow B$ Adams spectral sequence [Mat14a, Cor. 4.4] that (15) degenerates (for $t - s \geq 0$) after a finite stage with a horizontal vanishing line.

Remark 7.6. We refer to [MS14] for several computational examples and applications of this spectral sequence. In this paper, we will only use the existence of this spectral sequence and its degeneration at a finite stage with a horizontal vanishing line.

7.2. The main result.

Theorem 7.7. *Let A be a rational, noetherian \mathbf{E}_∞ -ring. Then the cokernel of the map $\mathrm{Pic}(\pi_*(A)) \rightarrow \mathrm{Pic}(A)$ is torsion-free.*

The main goal of this subsection is to prove Theorem 7.7, which while not entirely satisfying still provides significant information. In other settings, most of the interesting information in such Picard groups is precisely the torsion. We will prove this in several steps, following the construction of residue fields in Theorem 4.14.

Lemma 7.8. *Let A be a rational \mathbf{E}_∞ -ring and let $y \in \pi_{-1}(A)$. Then the kernel of $\mathrm{Pic}(A) \rightarrow \mathrm{Pic}(A//y)$ is a \mathbb{Q} -vector space.*

Proof. In fact, $A \rightarrow A//y$ admits descent (Proposition 3.8), so we use the homotopy spectral sequence associated to the expression (14). We observe that, for $i \geq 2$, the cosimplicial abelian group $\pi_i(\mathrm{pic}(B^{\otimes \bullet+1}))$ consists of rational vector spaces. Therefore, to prove the lemma, it suffices to show that there is no contribution in filtration one. However, we note that $\pi_0((A//y)^{\otimes k}) \simeq \pi_0(A//y)[u_1, \dots, u_{k-1}]$. As a result, the cosimplicial abelian group $\pi_0((A//y)^{\otimes \bullet+1})^\times$ is actually constant and has no nontrivial cohomology. This proves the claim. \square

Lemma 7.9. *Let A be a rational \mathbf{E}_∞ -ring such that:*

- (1) $\pi_2(A)$ contains a unit.
- (2) $\pi_0(A)$ is a field k .
- (3) $\pi_{-1}(A)$ is countably dimensional vector space over k .

Then the torsion subgroup of $\mathrm{Pic}(A)$ is $\{A, \Sigma A\}$.

Proof. We will imitate the argument of Proposition 4.11. Consider the sequence of \mathbf{E}_∞ -rings $A \simeq A^{(0)} \rightarrow A^{(1)} \rightarrow A^{(2)} \rightarrow \dots$ of Proposition 4.9. By Lemma 7.8, the maps

$$\mathrm{Pic}(A^{(i)})_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A^{(i+1)})_{\mathrm{tors}}$$

are injections.

Let $A_1 = \varinjlim A^{(i)}$. It follows that the map $A \rightarrow A_1$ induces an injection $\mathrm{Pic}(A)_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A_1)_{\mathrm{tors}}$, because the Picard functor commutes with filtered colimits [MS14, Prop. 2.4.1]. Moreover, A_1 satisfies the same hypotheses, and we can construct a sequence of \mathbf{E}_∞ -rings $A_1 \simeq A_1^{(0)} \rightarrow A_1^{(1)} \rightarrow \dots$ from Proposition 4.9. Similarly, each of the maps $A_1^{(i)} \rightarrow A_1^{(i+1)}$ induces an injection $\mathrm{Pic}(A_1^{(i)})_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A_1^{(i+1)})_{\mathrm{tors}}$. If we set $A_2 = \varinjlim A_1^{(i)}$, we get that the map $A_1 \rightarrow A_2$ induces an injection $\mathrm{Pic}(A_1)_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A_2)_{\mathrm{tors}}$. Inductively, we follow the proof of Proposition 4.11 and construct the sequence

$$A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots,$$

such that each A_i satisfies the same hypotheses as A did, and such that $A_i \rightarrow A_{i+1}$ is a colimit of a sequence constructed in Proposition 4.9, whose colimit is $k[t_2^{\pm 1}]$. Our reasoning shows that we get a sequence of injections

$$\mathrm{Pic}(A)_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A_1)_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A_2)_{\mathrm{tors}} \rightarrow \dots \rightarrow \mathrm{Pic}(k[t_2^{\pm 1}])_{\mathrm{tors}} \simeq \mathbb{Z}/2$$

because, again, the Picard functor commutes with filtered colimits. This completes the proof. \square

Lemma 7.10. *Let A be a rational, noetherian \mathbf{E}_∞ -ring. Suppose that $\pi_2(A)$ contains a unit and that $\pi_0(A)$ is a local artinian ring with residue field. Let $x \in \pi_0(A)$ belong to the maximal ideal. Then the map $A \rightarrow A//x$ induces an injection $\mathrm{Pic}(A)_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A//x)_{\mathrm{tors}}$.*

Proof. The map $A \rightarrow A//x$ admits descent since x is nilpotent (Proposition 3.7). Therefore, we can apply the expression (14) and the associated spectral sequence. As in Lemma 7.9, to run the argument, it suffices to see that there are no torsion contributions in filtration one. All the rings $\pi_0((A//x)^{\otimes(n+1)})$ are local artinian with the same residue field k , in view of Theorem 2.22. Given any local artinian ring R with residue field k , the group R^\times of units has a natural splitting $k^\times \oplus R^{\times, \mathrm{unip}}$ where $R^{\times, \mathrm{unip}}$ is a \mathbb{Q} -vector space. From this, it follows easily that the contribution in filtration one in the spectral sequence is a \mathbb{Q} -vector space, which proves the lemma. \square

Lemma 7.11. *Let A be a rational, noetherian \mathbf{E}_∞ -ring. Suppose that $\pi_2(A)$ contains a unit and that $\pi_0(A)$ is a local artinian ring. Then the only nontrivial element in the torsion subgroup of $\mathrm{Pic}(A)$ is ΣA .*

Proof. As in the proof of Proposition 4.12, we can construct a sequence of \mathbf{E}_∞ -rings

$$A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots,$$

such that:

- (1) $A_{i+1} \simeq A_i / (x_1^{(i)}, \dots, x_{n_i}^{(i)})$ for some finite sequence of nilpotent elements $x_j^{(i)} \in \pi_0(A_i)$.
- (2) Each ring $\pi_0(A_i)$ is local artinian with residue field k .
- (3) Each map $A_i \rightarrow A_{i+1}$ induces a map of local artinian rings $\pi_0(A_i) \rightarrow \pi_0(A_{i+1})$ which annihilates the maximal ideal of $\pi_0(A_i)$.

By Lemma 7.10, it follows that each of the maps $A_i \rightarrow A_{i+1}$ induces an injection $\mathrm{Pic}(A_i)_{\mathrm{tors}} \rightarrow \mathrm{Pic}(A_{i+1})_{\mathrm{tors}}$. The colimit $A_\infty = \varinjlim_i A_i$ has $\mathrm{Pic}(A_\infty)_{\mathrm{tors}} \simeq \mathbb{Z}/2$ in view of Lemma 7.9, and $\mathrm{Pic}(A)_{\mathrm{tors}} \hookrightarrow \mathrm{Pic}(A_\infty)_{\mathrm{tors}}$. Putting everything together, we get that $\mathrm{Pic}(A)_{\mathrm{tors}} \simeq \mathbb{Z}/2$ as desired. \square

Proof of Theorem 7.7. Let M be an invertible A -module. Suppose that $n > 0$ and that the tensor power $M^{\otimes n} \in \mathrm{Pic}(A)$ has the property that $\pi_*(M)$ is a flat $\pi_*(A)$ -module, i.e., $M^{\otimes n}$ belongs to the image of the map $\mathrm{Pic}(\pi_*(A)) \rightarrow \mathrm{Pic}(A)$. We need to show that M itself has this property.

For this, we may make the base change $A \rightarrow A[t_2^{\pm 1}]$, as $\pi_*(A) \rightarrow \pi_*(A[t_2^{\pm 1}])$ is faithfully flat, and thus assume that $\pi_2(A)$ contains a unit. Since flatness is a local property, it also suffices to work under the assumption that $\pi_0(A)$ is a local ring. Similarly, by completing, we can assume that $\pi_0(A)$ is complete local with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n) \subset \pi_0(A)$. Here we have to use the fact that any invertible A -module is necessarily perfect [MS14, Prop. 2.1.2]. In this case, the algebraic Picard group is trivial (Proposition 7.3). So it suffices to show that $\mathrm{Pic}(A)_{\mathrm{tors}} = \mathbb{Z}/2$.

Consider the tower of \mathbf{E}_∞ - A -algebras $A_m \stackrel{\mathrm{def}}{=} A / (x_1^m, \dots, x_n^m)$. We have $\mathrm{Pic}(A_m)_{\mathrm{tors}} \simeq \mathbb{Z}/2$ by Lemma 7.11. One sees that $\mathrm{Pic}(A) \subset \pi_0(\varprojlim_m \mathrm{pic}(A_m))$ because $\mathrm{Mod}^\omega(A) \subset \varprojlim_m \mathrm{Mod}^\omega(A_m)$. Note that the tower of abelian groups $\{\pi_1(\mathrm{pic}(A_m))\} = \{\pi_0(A_m)^\times\}$ satisfies the Mittag-Leffler condition since the tower $\{\pi_0(A_m)\}$ of finite-dimensional k -vector spaces clearly satisfies the Mittag-Leffler condition. In particular, $\pi_0(\varprojlim_m \mathrm{pic}(A_m)) = \varprojlim_m \mathrm{Pic}(A_m)$ by the Milnor exact sequence. Finally,

$$\mathrm{Pic}(A)_{\mathrm{tors}} \subset \left(\varprojlim_m \mathrm{Pic}(A_m) \right)_{\mathrm{tors}} \subset \varprojlim_m \mathrm{Pic}(A_m)_{\mathrm{tors}} = \varprojlim_m \mathbb{Z}/2 = \mathbb{Z}/2.$$

\square

8. NON-NOETHERIAN COUNTEREXAMPLES

The purpose of this section is to describe certain counterexamples that can arise from non-noetherian rational \mathbf{E}_∞ -rings. In particular, we obtain as a result new constructions of Galois extensions of ring spectra (Theorem 8.15) and of elements in Picard groups (Example 8.17). The main point of this section is that we can obtain such examples for quasi-affine schemes (such as punctured spectra) which fail to satisfy a form of “purity.”

8.1. Quasi-affineness. Let X be a noetherian scheme. Recall the presentable, stable ∞ -category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves of \mathcal{O}_X -complexes on X [Lur11a]. $\mathrm{QCoh}(X)$ is a symmetric monoidal ∞ -category with unit the structure sheaf \mathcal{O}_X , whose endomorphisms are given by the \mathbf{E}_∞ -ring $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ of (derived) global sections of \mathcal{O}_X . In this generality, we obtain an adjunction

$$(16) \quad (\cdot \otimes_{\mathbf{R}\Gamma(X, \mathcal{O}_X)} \mathcal{O}_X, \mathbf{R}\Gamma): \mathrm{Mod}(\mathbf{R}\Gamma(X, \mathcal{O}_X)) \rightleftarrows \mathrm{QCoh}(X),$$

which sends the $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ -module $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ to the structure sheaf \mathcal{O}_X . The right adjoint takes the derived global sections. We need the following basic fact.

Theorem 8.1 ([Lur11a, Prop. 2.4.4]). *Suppose X is quasi-affine. Then the adjunction (16) is a pair of inverse equivalences of symmetric monoidal ∞ -categories.*

\mathbf{E}_∞ -rings of the form $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ for quasi-affine schemes will be our primary source of counterexamples, because questions about $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ can often be reduced to questions (in ordinary algebraic geometry) about the scheme X . For instance, we obtain immediately:

Corollary 8.2. *Suppose X is a quasi-affine scheme and \mathcal{L} is a line bundle on X . Then the $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ -module $\mathbf{R}\Gamma(X, \mathcal{L})$ is invertible.*

We can also obtain a comparison for Galois theory.

Corollary 8.3. *Suppose X is a quasi-affine scheme. Suppose $Y \rightarrow X$ is a finite étale cover. Then the map $\mathbf{R}\Gamma(X, \mathcal{O}_X) \rightarrow \mathbf{R}\Gamma(Y, \mathcal{O}_Y)$ of \mathbf{E}_∞ -rings is a finite cover. If $Y \rightarrow X$ is a G -torsor for a finite group G , then the map (together with the natural G -action on the target) exhibits $\mathbf{R}\Gamma(Y, \mathcal{O}_Y)$ as a faithful G -Galois extension of $\mathbf{R}\Gamma(X, \mathcal{O}_X)$. In fact, the Galois group of the \mathbf{E}_∞ -ring $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ is naturally identified with the étale fundamental group of X .*

Proof. If $f: Y \rightarrow X$ is a finite étale cover, then Y can be constructed as the relative Spec of a sheaf of commutative \mathcal{O}_X -algebras $f_*(\mathcal{O}_Y)$. On any affine open $\mathrm{Spec} R \subset X$, $f_*(\mathcal{O}_Y)|_{\mathrm{Spec} R}$ is obtained from a finite étale algebra. In other words, if we write $\mathrm{QCoh}(X) = \varprojlim_{\mathrm{Spec} R \subset X} \mathrm{Mod}(R)$, as the inverse limit ranges over all Zariski open subsets $\mathrm{Spec} R \subset X$, then $f_*(\mathcal{O}_Y) \in \mathrm{QCoh}(X)$ defines a family of finite covers in each of these symmetric monoidal ∞ -categories. It follows from [Mat14a, Prop. 7.1] (and [Mat14a, Th. 6.5] for the comparison between weak finite covers and finite covers) that $f_*(\mathcal{O}_Y)$ defines a finite cover of the unit in the ∞ -category $\mathrm{QCoh}(X)$. Thus, applying the equivalence $\mathbf{R}\Gamma$ then completes the proof. \square

In general, there is no reason for the Galois theory (resp. Picard group) of the quasi-affine scheme to be determined by that of $\pi_0 \mathbf{R}\Gamma(X, \mathcal{O}_X)$, and this will be a source of counterexamples. This is related to subtle purity questions. However, we can obtain immediately counterexamples to the thick subcategory theorem without noetherian hypotheses in view of the following result.

Theorem 8.4 (Thomason [Tho97, Th. 3.15]). *Let X be a noetherian scheme. Let $\mathrm{QCoh}^\omega(X)$ denote the ∞ -category of quasi-coherent complexes $\mathcal{F} \in \mathrm{QCoh}(X)$ such that for every open affine $\mathrm{Spec} R \subset X$, $\mathcal{F}(\mathrm{Spec} R)$ is a perfect R -module (equivalently, $\mathrm{QCoh}^\omega(X)$ consists of the dualizable objects in $\mathrm{QCoh}(X)$). Then the thick tensor-ideals in $\mathrm{QCoh}^\omega(X)$ are in natural bijection with the specialization-closed subsets of X .*

Suppose X is quasi-affine. In this case, $\mathrm{QCoh}^\omega(X)$ corresponds under the equivalence of Theorem 8.1 to the $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ -modules which are dualizable, i.e., perfect, and thick tensor-ideals are the same as thick subcategories. Thus, we get:

Corollary 8.5. *If X is a quasi-affine scheme, then the thick subcategories of $\mathrm{Mod}^\omega(\mathbf{R}\Gamma(X, \mathcal{O}_X))$ are in natural bijection with the specialization-closed subsets of X .*

Construction 8.6. Let X be a quasi-affine, noetherian scheme which is not affine. In this case, the natural map $X \rightarrow \mathrm{Spec} \pi_0(\mathbf{R}\Gamma(X, \mathcal{O}_X))$ is an open immersion of schemes [Sta13, Tag 01P5]. By Corollary 8.5, the thick subcategories of $\mathrm{Mod}(\mathbf{R}\Gamma(X, \mathcal{O}_X))$ are in bijection not with specialization-closed subsets of the scheme $\mathrm{Spec} \pi_0(\mathbf{R}\Gamma(X, \mathcal{O}_X))$, but rather specialization-closed subsets of an open subset (i.e., X) of this scheme.

8.2. The punctured affine plane. Not every compact \mathbf{E}_∞ - \mathbb{Q} -algebra has the noetherianness properties used in this paper. In this subsection, we explain how \mathbf{E}_∞ -rings of the form $\mathbf{R}\Gamma(X, \mathcal{O}_X)$ can give counterexamples, and work out the simplest nontrivial case.

Construction 8.7. Consider the \mathbf{E}_∞ -ring A of functions on the punctured affine plane $\mathbb{A}^2 \setminus \{(0, 0)\}$, which fits into a homotopy pullback

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{Q}[x^{\pm 1}, y] \\ \downarrow & & \downarrow \\ \mathbb{Q}[x, y^{\pm 1}] & \longrightarrow & \mathbb{Q}[x^{\pm 1}, y^{\pm 1}] \end{array} .$$

The homotopy groups $\pi_*(A)$ are given by

$$\pi_i(A) = \begin{cases} \mathbb{Q}[x, y] & i = 0 \\ \mathbb{Q}[x, y]/(x^\infty, y^\infty) & i = -1 \\ 0 & \text{otherwise} \end{cases},$$

where $\mathbb{Q}[x, y]/(x^\infty, y^\infty)$ denotes the cokernel of the map $\mathbb{Q}[x^{\pm 1}, y] \oplus \mathbb{Q}[x, y^{\pm 1}] \rightarrow \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. In particular, $\pi_{-1}(A)$ is not a finitely generated $\pi_0(A)$ -module (though $\pi_0(A)$ is noetherian).

Proposition 8.8. *The \mathbf{E}_∞ -ring A is compact in $\mathrm{CAlg}_{\mathbb{Q}/}$.*

We are grateful to J. Lurie for explaining this to us.

Proof. In fact, to give a morphism $A \rightarrow B$, for B a rational \mathbf{E}_∞ -ring, is equivalent to giving two elements $u, v \in \Omega^\infty B$ which have the *property* that $B/(x, y)$ is contractible. This follows from the fact that A is the *finite localization* ([Mil92]) of $\mathbb{Q}[x, y]$ away from the $\mathbb{Q}[x, y]$ -module $\mathbb{Q}[x, y]/(x, y)$ (which is supported at the origin). In particular, to give a morphism of \mathbf{E}_∞ -rings $A \rightarrow B$ is equivalent to giving a map $\mathbb{Q}[x, y] \rightarrow B$ such that $B/(x, y) = B/(x, y)$ is contractible; note that this condition is detected in a finite stage of a filtered colimit. \square

Forthcoming work of B. Bhatt and D. Halpern-Leinster gives in fact an *explicit presentation* of A as an \mathbf{E}_∞ -ring under $\mathbb{Q}[x, y]$. Consider the $\mathbb{Q}[x, y]$ -module $M = \mathbb{Q}[x, y]/(x, y)$ and the natural map $\phi: \mathbb{Q}[x, y] \rightarrow M$. The dual gives a map $\psi: \mathbb{D}M \rightarrow \mathbb{Q}[x, y]$, where $\mathbb{D}M$ is the Spanier-Whitehead dual of M . Then one has:

Proposition 8.9 (Bhatt, Halpern-Leinster). *The \mathbf{E}_∞ - $\mathbb{Q}[x, y]$ -algebra is the pushout*

$$\begin{array}{ccc} \mathrm{Sym}_{\mathbb{Q}[x, y]}^*(\mathbb{D}M) & \longrightarrow & \mathbb{Q}[x, y] \\ \downarrow & & \downarrow \\ \mathbb{Q}[x, y] & \longrightarrow & A \end{array}$$

where:

- (1) $\mathrm{Sym}_{\mathbb{Q}[x, y]}^*(\mathbb{D}M)$ is the free \mathbf{E}_∞ - $\mathbb{Q}[x, y]$ -algebra on the $\mathbb{Q}[x, y]$ -module $\mathbb{D}M$.
- (2) The two maps $\mathrm{Sym}_{\mathbb{Q}[x, y]}^*(\mathbb{D}M) \rightarrow \mathbb{Q}[x, y]$ are adjoint to two maps of $\mathbb{Q}[x, y]$ -modules $\mathbb{D}M \rightarrow \mathbb{Q}[x, y]$ which are given by ψ and the zero map.

Proof. Indeed, let $A' \in \mathrm{CAlg}_{\mathbb{Q}[x, y]/}$. Then

$$\mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{Q}[x, y]/}}(A, A') \simeq * \times_{\mathrm{Hom}_{\mathrm{Mod}(\mathbb{Q}[x, y])}(\mathbb{D}M, A')} *.$$

Here the two maps $* \rightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathbb{Q}[x, y])}(\mathbb{D}M, A')$ send, respectively, $*$ to 0 and to the map $\mathbb{D}M \xrightarrow{\psi} A \rightarrow A'$. If $A'/(x, y) = 0$, then $\mathbb{D}M = 0$ and the mapping space $\mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{Q}[x, y]/}}(A, A')$ is therefore contractible. If not, then $\mathbb{D}M \rightarrow A \rightarrow A'$ is not the zero map, so that mapping space is empty. This is precisely the universal property of $A' \in \mathrm{CAlg}_{A'/}$. \square

Remark 8.10. The ∞ -category of A -modules is equivalent to the ∞ -category of quasi-coherent sheaves on the scheme $\mathbb{A}_{\mathbb{Q}}^2 \setminus \{(0, 0)\}$, since this scheme is quasi-affine. In particular, it follows Corollary 8.5 that the thick subcategories of $\mathrm{Mod}^\omega(A)$ correspond to the subsets of $\mathbb{A}_{\mathbb{Q}}^2 \setminus \{(0, 0)\}$ which are closed under specialization. In particular, Theorem 1.4 fails for A , as there is no thick subcategory corresponding to the origin in $\mathrm{Spec} \pi_0 A$.

Remark 8.11. A also illustrates the failure of Theorem 2.22 in the non-noetherian case. In fact, $\pi_0(A)/(x, y) \simeq \mathbb{Q}$ while $A/(x, y)$ is contractible.

8.3. Punctured spectra and counterexamples. We will now describe counterexamples to our theorems on Galois groups and Picard groups in the non-noetherian case, arising from quasi-affine schemes in a similar way. We start by recalling the context.

Construction 8.12. Let (R, \mathfrak{m}) be a noetherian local ring. We define the *punctured spectrum* $\mathrm{Spec}^\circ R = \mathrm{Spec} R \setminus \{\mathfrak{m}\}$.

The punctured spectrum $\mathrm{Spec}^\circ R$ is a quasi-affine scheme, and many “purity” results in algebraic geometry and commutative algebra relate invariants of $\mathrm{Spec}^\circ R$ to those of $\mathrm{Spec} R$.

Theorem 8.13 (Zariski-Nagata [Gro05, Exp. X, Th. 3.4]). *Let R be a regular local ring of dimension ≥ 2 . Then the inclusion $\mathrm{Spec}^\circ R \rightarrow \mathrm{Spec} R$ induces an isomorphism on étale fundamental groups.*

From our point of view, we can restate “purity” results such as the Zariski-Nagata theorem in terms of ring spectra, by passage to the \mathbf{E}_∞ -ring $\mathbf{R}\Gamma(\mathrm{Spec}^\circ R, \mathcal{O}_{\mathrm{Spec}^\circ R})$. Let (R, \mathfrak{m}) be a regular local ring of dimension ≥ 2 . Then $\pi_0(\mathbf{R}\Gamma(\mathrm{Spec}^\circ R, \mathcal{O}_{\mathrm{Spec}^\circ R})) = R$ and $\pi_{-i}(\mathbf{R}\Gamma(\mathrm{Spec}^\circ R, \mathcal{O}_{\mathrm{Spec}^\circ R})) = 0$ if $i \notin \{0, \dim(R) - 1\}$ by general results on local cohomology and depth [Gro05, Exp. III, Ex. 3.4]. For example, we obtain:

- (1) Theorem 8.13 is thus equivalent to the statement that the Galois group of the \mathbf{E}_∞ -ring $\mathbf{R}\Gamma(\mathrm{Spec}^\circ R, \mathcal{O}_{\mathrm{Spec}^\circ R})$ is algebraic.
- (2) Similarly, on a much more elementary level, let (R, \mathfrak{m}) be a regular local ring of dimension ≥ 2 . Then R is factorial, so that it has trivial Picard group. Since the Picard group is isomorphic to the class group, it follows that the inclusion $\mathrm{Spec}^\circ R \rightarrow \mathrm{Spec} R$ induces an isomorphism on Picard groups. In particular, it follows that the Picard group of $\mathbf{R}\Gamma(\mathrm{Spec}^\circ R, \mathcal{O}_{\mathrm{Spec}^\circ R})$ is algebraic. (More subtle purity results of the Picard group in non-regular cases can be phrased in this form too.)

The main point of this subsection is that non-regular rings for which purity fails can be used to give interesting examples of Galois extensions and invertible modules over non-noetherian ring spectra. Our example (which is not local) follows [Fos73, Example 16.5].

Let K be a field of characteristic zero containing a primitive n th root ζ_n of unity. Let $m \geq 2$. Consider the \mathbb{Z}/n -action on the ring $R' = K[x_1, \dots, x_m]$ sending $x_i \mapsto \zeta_n x_i$. Then $R = R'^{\mathbb{Z}/n}$ is the subring generated by all the homogeneous degree n monomials. Geometrically, the map $\mathrm{Spec} R' \rightarrow \mathrm{Spec} R = \mathrm{Spec} R'^{\mathbb{Z}/n}$ corresponds to the quotient of the affine space \mathbb{A}^m by rotation by the angle $2\pi/n$ in each direction. In particular, this map is étale away from the origin, the only place where the action fails to be free.

Construction 8.14. Let $X = \mathrm{Spec} R$ and let $Y = \mathrm{Spec} R'$. We have a \mathbb{Z}/n -action on Y and a map $Y \rightarrow X$ which exhibits X as the quotient $Y/(\mathbb{Z}/n)$. If $y \in Y$ is the point corresponding to the prime ideal (x_1, \dots, x_m) and $x \in X$ its image, then the point y is \mathbb{Z}/n -invariant, and the induced map $Y \setminus \{y\} \rightarrow X \setminus \{x\}$ is a \mathbb{Z}/n -torsor. We write $X^\circ = X \setminus \{x\}$, $Y^\circ = Y \setminus \{y\}$.

We define \mathbf{E}_∞ -rings $A = \mathbf{R}\Gamma(X^\circ, \mathcal{O}_{X^\circ})$ and $B = \mathbf{R}\Gamma(Y^\circ, \mathcal{O}_{Y^\circ})$. Note that $B \in \mathrm{CAlg}_{A/}$ has a natural \mathbb{Z}/n -action.

Theorem 8.15. *We have $\pi_0(A) = R$, $\pi_0(B) = R'$. The map $A \rightarrow B$, together with the \mathbb{Z}/n -action on B exhibits B as a faithful \mathbb{Z}/n -Galois extension of A .*

Proof. The natural map $R' = \Gamma(Y, \mathcal{O}_Y) \rightarrow \Gamma(Y^\circ, \mathcal{O}_{Y^\circ})$ is an isomorphism since Y is normal and the missing locus is codimension ≥ 2 by [Mat80, §17, Th. 35]. Moreover, the \mathbb{Z}/n -torsor $Y^\circ \rightarrow X^\circ$ shows that $\mathbf{R}\Gamma(X^\circ, \mathcal{O}_{X^\circ}) \simeq \mathbf{R}\Gamma(Y^\circ, \mathcal{O}_{Y^\circ})^{h\mathbb{Z}/n}$. Taking π_0 , we find that $\pi_0(A) \simeq (\pi_0(\Gamma(Y^\circ, \mathcal{O}_{Y^\circ})))^{\mathbb{Z}/n} = R'^{\mathbb{Z}/n} = R$. The assertion that $A \rightarrow B$ is a faithful \mathbb{Z}/n -Galois extension comes from Corollary 8.3. \square

The map of commutative rings $R \rightarrow R'$ is not étale: in fact, R' is not regular (at zero) while R is. In particular, the Galois extension of Theorem 8.15 does not come from algebra.

Example 8.16. Suppose $K = \mathbb{C}$ is the field of complex numbers. In this case, the topological realization of $\mathrm{Spec} R$ (i.e., the topological space $\mathbb{C}^m/(\mathbb{Z}/n)$) is easily seen to be *contractible*: one

can scale down to the image of the origin. Therefore, the étale fundamental group of $R = \pi_0(A)$ is trivial. However, the \mathbb{Z}/n -Galois extension $A \rightarrow B$ (and the fact that B has trivial Galois group by Corollary 8.3 applied to $\mathbb{C}^m \setminus \{(0, \dots, 0)\}$) shows that the Galois group of the \mathbf{E}_∞ -ring A is *precisely* \mathbb{Z}/n .

We can also obtain elements of the Picard group.

Example 8.17. We compute the (classical) Picard group of the scheme X° again with $K = \mathbb{C}$. First, we observe that the Picard group of Y° is trivial since that of affine space Y is and $Y^\circ = Y \setminus \{y\}$. Moreover, $\Gamma(Y^\circ, \mathcal{O}_{Y^\circ}) = \mathbb{C}^\times$. We have a \mathbb{Z}/n -torsor $Y^\circ \rightarrow X^\circ$, and we can use Galois descent to compute the Picard group as $\text{Pic}(X^\circ) = H^1(\mathbb{Z}/n; H^0(Y^\circ, \mathcal{O}_{Y^\circ}^\times)) = \mathbb{Z}/n$ since the action is trivial. Therefore, the Picard group of the \mathbf{E}_∞ -ring A is given by $\mathbb{Z} \oplus \mathbb{Z}/n$ where the \mathbb{Z} comes from suspensions. However, we claim that the Picard group of $\pi_0(A)$ is trivial. Indeed, by Lemma 8.18 below, we have $\text{Pic}(X) \subset \text{Pic}(X^\circ) = \mathbb{Z}/n$. But the Picard group of X can have no torsion as X is topologically contractible, in view of the K ummer sequence, and therefore $\text{Pic}(X) = 0$. In particular, by Corollary 8.2, we find that $\text{Pic}(A) \simeq \mathbb{Z} \oplus \mathbb{Z}/n$ though the Picard group of $\pi_0(A)$ is trivial.

Lemma 8.18. *Let X be a noetherian, normal, integral scheme. Let $Z \subset X$ have codimension ≥ 2 . Then the map $\text{Pic}(X) \rightarrow \text{Pic}(X \setminus Z)$ is injective.*

Proof. Let $j: X \setminus Z \rightarrow X$ be the open imbedding. Then the map $\mathcal{O}_X^\times \rightarrow j_*(\mathcal{O}_{X \setminus Z}^\times)$ is an isomorphism. The Leray spectral sequence now shows that the natural map $H^1(X, \mathcal{O}_X^\times) \rightarrow H^1(X \setminus Z, \mathcal{O}_{X \setminus Z}^\times)$ is an injection. \square

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UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840

E-mail address: amathew@math.berkeley.edu

URL: <http://math.berkeley.edu/~amathew>